Wed. - Week 13

Last Time: A homomorphism $f: G \longrightarrow \bar{G}$ preserves the id, $f(e) = \bar{e}$.

proof: $f(e)$ is something in $\bar{G}$, say $f(e) = \bar{g}$.

For any $g \in G$, $f(g) = f(g \cdot e) = f(g) f(e)$

$$f(g) = f(g) \cdot f(e)$$

apply the inverse of $f(g)$

$$\underbrace{[f(g)]^{-1} f(g)}_{\bar{e}} = \underbrace{[f(g)]^{-1} f(g)}_{\bar{e}} f(e)$$

$$\cdot f(e)$$

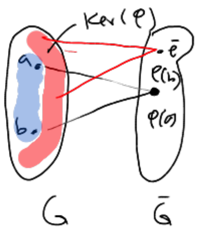
$$\bar{e} = \bar{e} \cdot f(e) = f(e).$$

Today: homomorphisms & 1st Isomorphism Theorem

Property of homom.:

Let $\varphi : G \to \bar{G}$ be a homom.

$$\varphi(a) = \varphi(b) \iff a\,\mathrm{Ker}(\varphi) = b\,\mathrm{Ker}(\varphi)$$




proof: $\boxed{\Rightarrow}$ Assume $\varphi(a) = \varphi(b)$

Show $a\,\mathrm{Ker}(\varphi) = b\,\mathrm{Ker}(\varphi)$

equiv. $ak_1 = bk_2$ for some $k_i \in \mathrm{Ker}$

show $\boxed{a = bk}$ for some $k \in \mathrm{Ker}$.

$\iff a = kb$ ✓

recall

$e = a \cdot a^{-1}$ in $G$

apply

$$\varphi(e) = \varphi(a \cdot a^{-1}) \underset{\text{homom.}}{=} \varphi(a)\varphi(a^{-1}) \underset{\text{assumption}}{=} \varphi(b)\cdot\varphi(a^{-1}) = \varphi(ba^{-1})$$

earlier — $\varphi(e) = \bar{e}$

So $\bar{e} = \varphi(ba^{-1})$

$\Rightarrow ba^{-1} \in \mathrm{Ker}(\varphi)$ So $ba^{-1} = k$ for some $k \in \mathrm{Ker}(\varphi)$

or

$b = ka$

$k^{-1}b = a$

( $k^{-1} \in \mathrm{Ker}(\varphi)$ b/c subgroup) since $\mathrm{Ker}(\varphi)$ is normal

$a\,\mathrm{Ker}(\varphi) = \mathrm{Ker}(\varphi)\,a$ ✓

$\Leftarrow$ If $\underbrace{a\,\mathrm{Ker}(\varphi) = b\,\mathrm{Ker}(\varphi)}$ then show $\varphi(a) = \varphi(b)$

we know that $a = bk$

$$\varphi(a) = \varphi(bk) = \varphi(b)\overbrace{\varphi(k)}^{\bar{e}} = \varphi(b)\cdot\bar{e} = \varphi(b)$$