

Wed. Week 3.

1. HW due Sat. a.m.

## 2. Properties of Groups

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1. The identity is unique.

Proof: If  $e_1, e_2$  are both identity elements in  $G$  then for any  $a \in G$ .

$$a \cdot e_1 = a = e_1 \cdot a \quad (i)$$

$$a \cdot e_2 = a = e_2 \cdot a \quad (ii)$$

(i) works for  $e_2 = a$  :  $e_2 \cdot e_1 = e_2$

(ii) works for  $e_1 = a$  :  $e_1 = e_2 \cdot e_1$

$$\text{So, } e_1 = e_2.$$

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2. Every elt. has a unique inverse.

Proof: If  $a_1, a_2$  are both inverses of  $a$ , then

$$a_1 \cdot (a \cdot a_2) = a_1 \cdot e = a_1$$

|| assoc:

$$(a_1 \cdot a) \cdot a_2 = e \cdot a_2 = a_2$$

$$\text{so } a_1 = a_2$$

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3. Cancellation Law.

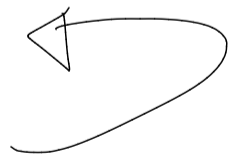
$$ab = cb \xrightarrow{\text{implies}} a = c$$

$$ba = bc \xrightarrow{\text{implies}} a = c$$

If  $ab = cb$ , then

$$\begin{aligned} ab \cdot b^{-1} &= cb \cdot b^{-1} \\ a \cdot bb^{-1} &= c \cdot bb^{-1} \\ a \cdot e &= c \cdot e \\ a &= c \end{aligned}$$

Same for



4. If  $ab = e$  then  $a = b^{-1}$ ,  $b = a^{-1}$ .

proof If  $ab = e$  then  $ab \cdot b^{-1} = e \cdot b^{-1}$  so  $a = b^{-1}$   
similarly for  $b = a^{-1}$   $\uparrow$   $\downarrow$   
rt rt

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5. Shoes & Socks Property!

image  $a =$  putting on socks,  $a^{-1} =$  take socks off  
 $b =$  putting on shoes,  $b^{-1} =$  take shoes off.

$b \cdot a$

$(b \cdot a)^{-1} =$  take off both socks/shoes

put on socks, THEN put on shoes  
(function way is right  $\rightarrow$  left)

FACT!  $(ba)^{-1} = a^{-1} \cdot b^{-1}$

(recall: Matrix M.H.  $A, B$  matrices

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

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6.  $(a^{-1})^{-1} = a$

proof: since  $a \cdot a^{-1} = e$  we know from property 4 that:  
so the inverse of  $a^{-1}$  is  $a$ , which means  $(a^{-1})^{-1} = a$ .

Six Problems in "Rules of Algebra"  
 Either prove or find counter example. (See:  $D_4$  or Matrix example w/ six elements)

1. If  $x^2 = e$  then  $x = e \quad \forall x \in G,$

{ $R_0, R_{90}, R_{180}, R_{270}, I, V, D, D'$ }

2. If  $x^2 = a^2$  then  $x = a \quad \forall x \in G$  ( $G = \mathbb{R}$ , mult  
 $(-1)^2 = (1)^2$  yet  $-1 \neq 1$ .)

3.  $(ab)^2 = a^2 b^2 \quad \forall a, b \in G$

4. If  $x^2 = x$  then  $x = e$  (Yes,

$x^2 = x \Rightarrow \underset{x}{x^2} \cdot \underset{x^{-1}}{x^{-1}} = \underset{e}{x \cdot x^{-1}}$ )

5.  $\forall x \in G, \exists y \in G$  s.t.  $x = y^2$ . (Every elt. in a group has a square root)

6. For any two elts  $x, y \in G, \exists z \in G$  s.t.  $x = yz$ .

5.  $\forall x \in G, \exists y \in G$  s.t.  $x = y^2$ . (Every elt. in a group has a square root)

Ex 4 has a square root:  $4 = 2^2$

Note!  $\mathbb{N} = \{0, 1, 2, \dots\}$   
id:  $= 0$   
inverse of 1:  $1 + \square = 0$  } Not a group

D4! Look @  $V \in D_4$ . How can I write  $V$  as the square of something?

#5 fails  $\forall v \in V \neq x^2$ . (not every elt has a sq. root)

#3/  $a = D$  |  $a^2 b^2 = D^2 V^2 = e \cdot e = e (= R_0)$  }  $\neq$   
 $b = V$  |  $DV = R_{90}$ ,  $(R_{90})^2 = R_{180}$

