Wed. Week 3.

1. HW due Sat. a.m.

2. Properties of Groups

1. The identity is unique.

Proof: If e_1, e_2 are both identity elements in G then for any $a \in G$.

$\underbrace{a \cdot e_1 = a}$ $= e_1 \cdot a$ (i)

$\&$ $a \cdot e_2 = a$ $\underbrace{= e_2 \cdot a}$, (ii)

(i) works for $e_2 = a$: $\overbrace{e_2 \cdot e_1} = e_2$

(ii) works for $e_1 = a$ $e_1 = \underbrace{e_2 \cdot e_1}$

So, $e_1 = e_2$.

2. Every elt. has a unique inverse.

Proof: If a_1 & a_2 are both inverses of a, then

$a_1 \cdot (\overbrace{a \cdot a_2}) = a_1 \cdot e = a_1$

$\|$ assoc.

$(a_1 \cdot a) \cdot a_2 = e \cdot a_2 = a_2$

so $a_1 = a_2$

3. Cancellation Law. $ab = cb \xRightarrow{\text{implies}} a = c$

&

$ba = bc \Rightarrow a = c$

If $ab = cb$, then $ab \cdot b^{-1} = cb \cdot b^{-1}$

$a \cdot bb^{-1} = c \cdot bb^{-1}$

$a \cdot e = c \cdot e$

$a = c$

Same for

4. If $ab = e$ then $a = b^{-1}$, $b = a^{-1}$.

proof If $ab = e$ then $ab \cdot b^{-1} = e \cdot b^{-1}$ so $a = b^{-1}$

rt rt

similarly for $b = a^{-1}$

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5. Shoes & Socks Property!

imagine a = putting on socks, a^{-1} = take socks off

b = putting on shoes b^{-1} = take shoes off.

$$\underbrace{b \cdot a}_{\text{put on socks, THEN put on shoes}}$$

(function way is right $\to$ left)

$(b \cdot a)^{-1}$ = take off both socks/shoes

FACT! $(ba)^{-1} = a^{-1} \cdot b^{-1}$

(recall: Matrix Mult. A, B matrices

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

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6. $(a^{-1})^{-1} = a$

proof: since $a \cdot a^{-1} = e$ we know from property 4 that:

so the inverse of a^{-1} is a, which means $(a^{-1})^{-1} = a$.

Six Problems in "Rules of Algebra"
Either prove or find counter example. (See: $\underline{D_4}$ or Matrix example w/ six elements)

$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$

1. If $x^2 = e$ then $x = e$ $\forall x \in G$.
2. If $x^2 = a^2$ then $x = a$ $\forall x \in G$ ($G = \mathbb{R}$, mult $(-1)^2 = (1)^2$ yet $-1 \neq 1$.
3. $(ab)^2 = a^2 b^2$ $\forall a, b \in G$
4. If $x^2 = x$ then $x = e$ (Yes, $x^2 = x \Rightarrow \underbrace{x^2 \cdot x^{-1}}_{= x} = \underbrace{x \cdot x^{-1}}_{e}$)
5. $\forall x \in G$, $\exists y \in G$ s.t. $x = y^2$. (Every elt. in a group has a square root)
6. For any two elts $x, y \in G$, $\exists z \in G$ s.t. $x = yz$.

5. $\forall x \in G$, $\exists y \in G$ s.t. $x = y^2$. (Every elt. in a group has a square root)

Ex 4 has a square root: $4 = 2^2$

Note: $\mathbb{N} = \{0, 1, 2, \ldots\}$
id: $= 0$
inverse of 1: $1 + \square = 0$

$\left.\right\}$ $\mathbb{N}$ not a group

D_4: Look @ $V \in D_4$. How can I write V as the square of something?

#5 fails b/c $V \neq x^2$. (Not every elt has a sq. root)

#3/ $a = D$, $b = V$

$a^2 b^2 = D^2 V^2 = e \cdot e = e \ (= R_0)$

$DV = R_{90}$, $(R_{90})^2 = R_{180}$

$\neq$

$$\begin{tikzcd} \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} \arrow[r, "V"] \arrow[rr, bend right, "R_{90}"'] & \begin{matrix} 2 & 1 \\ 4 & 3 \end{matrix} \arrow[r, "D"] & \begin{matrix} 2 & 4 \\ 1 & 3 \end{matrix} \end{tikzcd}$$