Week 5 Plan

1. Monday
   a. Basic properties of groups
   b. Center
   c. Centralizer
   d. Exercises from Ch. 3
2. Wednesday
   a. Cylic groups
3. Friday
   a. More cylic groups

Some basic properties of every group.

1. A 1-sided identity is really 2-sided.

(Def'n of group: you see "... $\forall a \in G$ $\exists$ e s.t. $\underbrace{a \cdot e}_{\text{right identity}} = a = \overbrace{e \cdot a}^{\text{left id}}$

Thm If $\forall a \in G$ $\exists$ e s.t. $a \cdot e = a$

proof: then $e \cdot a = a$.

Consider $e \cdot a$, it equals some elt. in G b/c of closure

$$e \cdot a = a'.$$

Hit w/ a on left:

$$\underbrace{a \cdot e}_{} \cdot a = a \cdot a'$$

$$a \cdot a = a \cdot a' \quad \text{by assumption.}$$

Hit w/ a^{-1} on left

$$\underbrace{a^{-1} a}_{e} \cdot a = a^{-1} a \, a'$$

$$\underset{a}{\underset{\parallel}{e a}} = e \cdot a' = a'$$

so $a = a'$

Similarly, a 1-sided inverse is two sided.

Center of group, denoted $Z(G)$

$$Z(G) = \{x \in G \mid xg = gx \ \forall x\ g \in G\}$$

the set of elements in G that commute with everything in G

Ex $G = (\mathbb{R}, +)$. $Z(G) = G$

Ex $G = D_4$. $Z(D_4) = \{e, R_{180}\}$

1. Any two rotations commute: $R_{90} R_{180} = R_{180} R_{90}$

b/c $R_{90} \cdot R_{90}^2 = R_{90}^2 \cdot R_{90}$

$R_{90} \cdot R_{90} \cdot R_{90} = R_{90} R_{90} R_{90}$

2. Let F = some reflection

R = rotation.

when does $RF = FR$?

RF is a reflection, $\therefore$ $RF = (RF)^{-1} = F^{-1} R^{-1} = FR^{-1}$

$\Longleftrightarrow$ $FR = FR^{-1}$

$R = R^{-1}$ $\Rightarrow$ $R = R_0$, or $R = R_{180}$

Ex: Matrix Mult.

$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$. $MA = M = AM$ ✓

$\Rightarrow$ $\boxed{\text{Fact: Center is a subgroup}}$

$M = -I$

$$M \cdot A = -I \cdot A = -A = -A \cdot I = A \cdot (-I) = A \cdot M$$

$N_1 = \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix} \Rightarrow NA = AN.$

$\boxed{N_1^{-1} = \begin{pmatrix} -1/x & 0 \\ 0 & -1/x \end{pmatrix} \text{ same form} \Rightarrow \text{closed under inverses}}$

$N_2 = \begin{pmatrix} -y & 0 \\ 0 & -y \end{pmatrix}$

$N_1 \cdot N_2 = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix} = \begin{pmatrix} -(-xy) & 0 \\ 0 & -(-xy) \end{pmatrix} \Rightarrow$ closed under $*$.

Thm: $Z(G) \leq G$.

proof:

1. Inverses: Let $a \in Z(G)$. Show $a^{-1} \in Z(G)$. $\left(\begin{array}{l}\text{show:} \\ a^{-1}b = ba^{-1} \\ \forall\, b \in G\end{array}\right)$

By assumption,

$$ab = ba \quad \forall\, b \in G$$

Hit w/ a^{-1} on right:

$$aba^{-1} = baa^{-1} = b$$

Now a^{-1} on left

$$\underbrace{a^{-1}a}_{e}ba^{-1} = a^{-1}b$$

$$ba^{-1} = a^{-1}b \qquad \text{QED}$$

2. closure under $\cdot$:

(show $\forall\, a,b \in Z(G)$, $ab \in Z(G)$)

If $a,b \in Z(G)$ =

$$\left(\begin{array}{ll} \text{B} & ax = xa \\ \text{A} & bx = xb \end{array} \quad \forall\, x \in G\right) \overset{\text{show}}{\Longrightarrow} abx = xab$$

By assumption $\quad abx \underset{\text{A}}{=} axb \underset{\text{B}}{=} xab \qquad \text{QED}$

Note:

Centralizer of an element:
set of elements that commute with a single fixed element.

$$C(g) = \{x \in G \mid gx = xg \text{ for a single fixed } g \in G\}$$

Ex: $G = D_4$

$$C(R_{90}) = \{\text{all rotations}\} = \{R_0, R_{90}, R_{270}, R_{180}\}$$

Thm: Centralizer is a subgroup.

$SL_2\mathbb{Z}$:

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$\sin(A+B) = \sin A\cos B + \sin B\cos A$

thm:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

proof by induction:

$n=1$ (trivial)

assume case n holds, prove $n+1$ case.

$$\begin{bmatrix} \cos(n+1)\theta & -\sin((n+1)\theta) \\ \sin(n+1)\theta & \cos(n+1)\theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{n+1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos n\theta & -\sin(n\theta) \\ \sin(n\theta) & \cos n\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cdot\cos n\theta - \sin\theta\sin n\theta & -\cos\theta\sin n\theta - \sin\theta\cos n\theta \\ \sin\theta\cos n\theta + \cos\theta\sin n\theta & -\sin\theta\sin n\theta + \cos\theta\cos n\theta \end{bmatrix}$$

$n=6$

$$\begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 6 & \cos 0 \end{bmatrix}^n = \begin{bmatrix} \cos(n60) & -\sin(n60) \\ \sin(n60) & \cos(n60) \end{bmatrix}$$

$$\begin{bmatrix} \cos\sqrt{2} & -\sin\sqrt{2} \\ \sin\sqrt{2} & \cos\sqrt{2} \end{bmatrix}^n \overset{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$