Monday – Week 6.

Homework # 4 – Due Next Monday (28^{th})

Permutation Groups:

A permutation of a set is:

1. function from the set to itself.
2. 1-1: If $f(a) = f(b)$ then $a = b$.

injective

3. onto: ($f: A \to B$ is onto if

surjective

$\forall\, b \in B\ \exists\ a \in A$ s.t. $f(a) = b$)

(Ex: $f(x) = x^3$ is 1-1.

if $(-a)^3 = (-b)^3$

$\Rightarrow -a = -b$

$f(x) = x^2$ is not 1-1.

$(-1)^2 = (1)^2$

yet $-1 \neq 1$

Ex: $f(x) = x^3$ is onto

every real # is hit by f.

If $r \in \mathbb{R}$ then $r = (r^{1/3})^3$

exists

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$ is not onto...

$-1 \neq x^2$ for any $x \in \mathbb{R}$.

We mainly focus on finite permutations

<u>Ex</u>: $S = \{0, 1, 2, 3, 4, 5, 6\}$

A permutation of S is, for example:

$\sigma_1(S) = \{6, 5, 4, 3, 2, 1, 0\}$

$\sigma_2(S) = \{0, 1, 2, 3, 4, 6, 5\}$

$$\begin{cases} \sigma_2(0) = 0 \\ \sigma_2(1) = 1 \\ \sigma_2(2) = 2 \\ \sigma_2(3) = 3 \end{cases} \qquad \begin{array}{l} \sigma_2(4) = 4 \\ \sigma_2(5) = 6 \\ \sigma_2(6) = 5 \end{array}$$

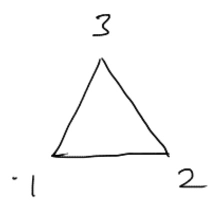
not $\sigma_3(S) = \{0, 0, 1, 2, 3, 4, 6\}$ (missed 5. (not onto))

Notation

one way to denote perms is:

$$\sigma_2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 6 & 5 \end{bmatrix}$$

<u>Ex</u>



write down all possible permutations of $\{1, 2, 3\}$

$$\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \qquad \beta_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \qquad e = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

This is S_3 = all possible permutations of 3 objects (usually these objects are 1, 2, 3)

6 different ways —— $6 = 3!$

In general,
S_n = symmetric group on n letters. $(1, 2, 3, \ldots n)$.

Fact:
If $\sigma \in S_n$ then σ has n choices of where to send 1,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \ldots & n \\ \downarrow & \downarrow & \downarrow & & \\ & n-1 & n-2 & & \end{bmatrix} \Rightarrow$$

$n-1$ choices

n choices

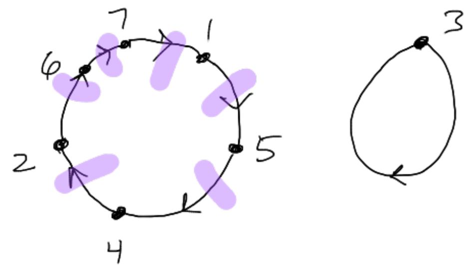
there are
$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$
elements in S_n

$|S_n| = n!$

Cycle Notation:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 3 & 2 & 4 & 7 & 1 \end{bmatrix}$$

Consider this permutation.



6-cycle

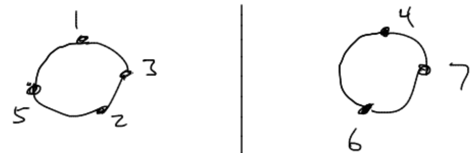
$\parallel$

$(1\,5\,4\,2\,6\,7)$

cycle notation

Now this one

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 2 & 7 & 1 & 4 & 6 \end{bmatrix} = \beta$$



4-cycle | 3-cycle

$(5\,1\,3\,2)$ $(6\,4\,7)$

$\beta = (5\,1\,3\,2)(6\,4\,7) = (6\,4\,7)(5\,1\,3\,2)$

b/c the cycles are disjoint!

Thm: Every permutation can be expressed as product of disjoint cycles.

Fact: A cycle is invariant under cyclic permutation.

$(5\,1\,3\,2) = (2\,5\,1\,3) = (1\,3\,2\,5) = (3\,2\,5\,1)$

Multiplying cycles

$(15)(14)(234)$ ← permutation on $\{1,2,3,4,5\}$

↑ start here

(14235)

Ex. $(15)(14)(13)(12) = (12345)$