Wednesday - Week 6

Symmetric Groups (Permutation Groups)

Let's Consider S_4 = symmetric group on 4 letters $(1,2,3,4)$

Elements of S_4:

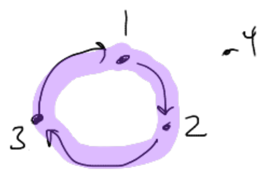
(1) = identity

(12)

$(12)(34)$

(123)

(1324)



Warm up:

multiply:

① $(12)(34)(123) = (1)(243) = (243)$

② $(1234)(12)(1423) = (24)$

Elements of S_4

$|S_4| = 4! = 24$

1	(1)
2	(12)
3	(23)
4	(34)
5	(13)
6	(14)
7	(24)
8	$(12)(34)$
9	$(13)(24)$
10	$(14)(23)$
11	(123)
12	(132)
13	(124)
14	(142)
15	(134)
16	(143)
17	(234)
18	(243)
19	(1234)
20	(1324)
21	(1423)
22	(1432)
23	(1243)
24	(1342)

In every S_n we define:

If $\pi \in S_n$ &

π can be written as a product of an even number of transpositions (2-cycles)

we say π is even.

Likewise:

$$\left.\begin{array}{c} \pi = t_1 \cdot t_2 \cdot \ldots \cdot t_{2k+1} \\ \text{odd \# of transposition} \end{array}\right\} \Longrightarrow \pi \text{ is odd}$$

Elements of S_4

1 (1)

2 (12)

3 (23)

4 (34)

5 (13)

6 (14)

7 (24)

each of these is a single (1) transposition $\Rightarrow$ odd

8 $(12)(34)$

9 $(13)(24)$

10 $(14)(23)$

even

11 $(123) = (13)(12)$

12 $(132) = (12)(13)$

13 $(124) = (14)(12)$

14 $(142) = (421) = (214) = (24)(21)$

15 (134)

16 (143)

17 (234)

18 (243)

even

odd's below

19 $(1\,2\,3\,4) = (14)(13)(12)$

20 $(1\,3\,2\,4)$

21 $(1\,4\,2\,3)$

22 $(1\,4\,3\,2)$

23 $(1\,2\,4\,3)$

24 $(1\,3\,4\,2)$

$$\underset{\substack{\parallel \\ (142)}}{(24)(21)} = \underset{\substack{\parallel \\ (142)}}{(12)(14)}$$

Also

$$(142) = (12)(14)\underbrace{(13)(13)}_{id}$$

① $\Rightarrow$ Any cycle can be written as a product of transpositions

② The number of transpositions may vary.

③ We'll see: an element can be written as either an even # of transpositions or odd # of "

– Not both.

Thm: the identity is an even permutation

Lemma: If we assume this logical statement:

If $e = t_1 t_2 \cdots t_m$ then $e = t_1 t_2 \cdots t_{m-2}$

t_i = transpositions

$\longmapsto m \longrightarrow$ $\quad$ $\longmapsto m-2$ transposit'.

Then $e \overset{=id}{}$ is even.

proof: If purple assumption holds and e is odd, then

$e = t_1 \cdot t_2 \cdots t_{13}$

apply purple assumption

$= t_1 \cdot t_2 \cdots t_{11}$

apply purple again

$= t_1 \cdot t_2 \cdots t_9$

$=$

$\vdots$

$e = t_1$ impossible since t_1 is a transposition (ab)

So our goal is to prove the purple cloud,

If $e = id$ then we can always reduce # of transpositions in its factoris by 2.

Assum $e = t_1 \cdot t_2 \cdots t_m$ ($e = (12)(32)(24)(41)(32)(12)$ t_{k-1} (42) t_k think $x = 4$)

let x = some numeral in $t_1 \cdot t_2 \cdots t_m$

let $t_k = (xa)$, and assume t_k is the last occurrence of

Now t_{k-1} is either:

I (xa) or II (xb) ✓ or III (ca), $c \neq x$ or IV (bc), $b \neq x, a$, $c \neq a, x$

$e = t_1 \cdot t_2 \cdots t_{k-1} t_k \cdots t_m$

$= t_1 t_2 \cdots \underbrace{(xa)(xa)}_{e} \cdots t_m = t_1 t_2 \cdots t_{m-2}$

II $e = t_1 \cdot t_2 \cdots t_{k-1} t_k \cdots t_m$

trick $(xb)(xa) = (xa)(ab)$

$= t_1 t_2 \cdot (xb)(xa) \cdots t_n$

$= t_1 t_2 \cdots (xa)(ab) \cdots t_n$

moved term w/ x to left.

similar for III, IV