Notes on Assignment 3

1. #22. Check carefully the assumption
2. If and only if requires two directions
3. Assignment 2 - some comments in side comment box (open in Kami)

G has the property that

$\underbrace{\text{if } xy = zx \text{ then } y = z}$,

Prove G is ablian.

The (errant) strategy of many:

★ Assume $xy = zx$.

let $x = a$

$y = ba$

$z = ab$

then $aba = aba$

assumption $\Rightarrow$

$y = z$

$ba = ab$.

Better: Start w/ the truth!

(1) $aba = aba$

then, let $x = a$

$y = ba$

$z = ab$

(2) $xy = zx$

(3) assumption on G $\Rightarrow$ $y = z$

$ba = ab$.

Isomorphisms

Properties of Isomorphisms on Elements

φ: homom, 1-1, onto

Let $\varphi : G \longrightarrow \overline{G}$ be an isomorphism.

1. φ maps the identity of G to the identity of $\overline{G}$.

proof

$$\varphi(e) = \varphi(e \cdot e) \overset{\text{homom}}{=} \varphi(e) \cdot \varphi(e)$$

e = id in G

$\overline{e}$ = id in $\overline{G}$

Note $\varphi(e) \in \overline{G}$

So ....

Ex. $\mathbb{Z}_4 = \{0, 1, 2, 3\}$

$H \leq D_4$, $H = \{R_0, R_{90}, R_{180}, R_{270}\}$

If $\varphi : \mathbb{Z}_4 \to H$ an isom, then

$\varphi(0) = R_0$

$$\underbrace{\varphi(e)^{-1}\varphi(e)}_{\text{id in } \overline{G}} = \underbrace{\varphi(e)^{-1}\varphi(e)}\cdot\varphi(e)$$

$$\overline{\overline{e}} = \overset{\text{id}}{\overline{e}} \cdot \varphi(e) = \varphi(e)$$

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2. $\varphi(a^n) = \varphi(a)^n \quad \forall \; n \in \mathbb{Z}$.

Ex $n = 2$

$$\varphi(a^2) = \varphi(a \cdot a) = \varphi(a) \cdot \varphi(a) = (\varphi(a))^2$$

By induction, true for $n \in \mathbb{Z}^+$.

If $n < 0$, i.e., $\varphi(a^{-1}) \overset{\text{why}}{=} \varphi(a)^{-1}$

then $-n > 0$ so,

b/c $-n > 0$ & use

$$\varphi(e) = \varphi(a^n \cdot a^{-n}) = \varphi(a^n) \cdot \varphi(a^{-n}) = \varphi(a^n) \cdot \varphi(a)^{-n}$$

" ①

$$\overline{e} = \varphi(a^n) \cdot \underbrace{\varphi(a)^{-n} \, \varphi(a)^n}_{\overline{e}}$$

cancellation

$\varphi(a)^n$

$$\overline{e} \cdot \varphi(a)^n = \varphi(a^n) \cdot \overline{e}$$

$$\varphi(a)^n = \varphi(a^n)$$

True $\forall \; n$

③ Iso's preserve commutivity.

Isomorphisms
1-1
onto
homom.

$$\varphi(ab) = \varphi(ba) \iff ab = ba$$

$\Rightarrow$ If $\varphi(ab) = \varphi(ba)$ then b/c φ is 1-1 $ab = ba$

$\Leftarrow$ If $ab = ba$ then $\varphi(ab) = \varphi(ba)$ b/c φ is a function.

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④ $G = \langle a \rangle \iff \bar{G} = \langle \varphi(a) \rangle$

G is cyclic generated by a

$\bar{G}$ cyclic generated by $\varphi(a)$.

$\Rightarrow$ If $G = \langle a \rangle$, then any element $g \in G$ looks like' $g = a^k$ for some $k \in \mathbb{Z}$.

If φ is an isom., $\varphi(a^k) = \varphi(a)^k \in \bar{G}$.

Does everything in $\bar{G}$ look like this?

let $b \in \bar{G}$ some random elt. in $\bar{G}$.

By property onto we know

$\forall\, b \in \bar{G}$ $\exists\, g \in G$ s.t. $\varphi(g) = b$

$$b = \varphi(g) = \varphi(a)^k,$$

So $\bar{G} = \langle \varphi(a) \rangle$

$\Leftarrow$ Assume $\bar{G} = \langle \varphi(a) \rangle$. Show $G = \langle a \rangle$.

Let $a \in G$, by def. of φ, $\varphi(a) \in \bar{G}$.

Let $b \in G$ be anything.

$\varphi(b) \in \bar{G}$ so

$\varphi(b) = \varphi(a)^k = \varphi(a^k)$ & φ is an isomorphism,

by assumption $\bar{G} = \langle \varphi(a) \rangle$.

because φ is 1-1

$b = a^k$.

$$\boxed{G = \langle a \rangle}$$

Isomorphisms preserve orders.

If $|a| = n$ $a^n = e$

& n is the smallest exp.

$\overset{\text{then}}{\Longrightarrow}$ $\varphi(a^n) = \varphi(a)^n = \overline{e}$

& n is the smallest exp.

$\boxed{\text{If and only If}}$

the statement P if and only if Q,

means

P if Q, i.e., if Q then P

$\boxed{Q \Rightarrow P}$

and

P only if Q,

If not Q, then not P. $(\sim Q \Rightarrow \sim P)$

by contrapositive

$\sim\sim P \Rightarrow \sim\sim Q$

$\boxed{P \Rightarrow Q}$

Ex. Show G abelian iff $(ab)^{-1} = a^{-1}b^{-1}$.

(if and only if)

Assume G abelian, show $(ab)^{-1} = a^{-1}b^{-1}$

Assume $(ab)^{-1} = a^{-1}b^{-1}$ show G is abelian