Happy Monday, Week 7 -

Note: Abstract Alg. II - Winter 20.

(Groups, Rings (2 operations), Division Rings, Fields) $\longrightarrow$

Note: Midterm! Next Wed.

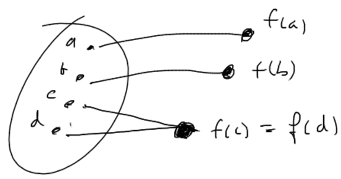
Today: Isomorphisms ———————————

when two groups are really the same.

First some reminders:

Def'n: 1-1, "one to one", A function $f: X \longrightarrow Y$ is 1-1 if:
(injective)

$$\boxed{\begin{array}{c} \text{If } f(c) = f(d) \\ \text{then} \\ c = d \end{array}}$$



Not 1-1

if the images of two elements are equal then the two elements themselves are equal

Def'n onto: A function $f: X \longrightarrow Y$ is onto if
(surjective)

$$\boxed{\forall\, y \in Y \quad \exists\, x \in X \quad \text{s.t. } f(x) = y.}$$

Def'n: A function $f: X \to Y$ is a homomorphism if:

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

Ex:

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

$$\lim_{x \to \infty}\left(f(x) + g(x)\right) = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x)$$

A, B are $n \times n$ matrices
$v \in n \times 1$ vector

Av, Bv are vectors:

$$(A+B)v = Av + Bv$$

Note: $f: X \longrightarrow Y$ means f is a function from x to y

domain range

Def'n An isomorphism f, from a group G to another group $\bar{G}$ is a (function) mapping from G to $\bar{G}$

that is:

1-1, onto, $\underbrace{\text{homomorphism}}_{\text{operation preserving}}$

<u>Note</u>: Every finite group is isomorphic to some subgroup of permutations

<u>Example of Isomorphisms</u>:

① $(\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, *)$. $f: \mathbb{R} \longrightarrow \mathbb{R}^+$, by $f(x) = e^x$.

$\longleftrightarrow$ $\quad$ $\circ\!\!\longrightarrow$

1-1: If $f(a) = f(b)$ then $e^a = e^b$ but then

$\ln(e^a) = \ln(e^b)$

$a = b$ ✓

onto: let $y \in \mathbb{R}^+$. Find some $x \in \mathbb{R}$ s.t.

$e^x = y$. (by solving this for x:

$x = \ln y$.

so $\ln y \in \mathbb{R}$ b/c $y \in \mathbb{R}^+$ thus

$x = \ln y$ is the element that is mapped onto y. ✓

homomorphism: operation preserving
show $f(a+b) = f(a) * f(b)$

$f(x) = e^x$, plug in $a+b$:

$f(a+b) = e^{a+b} = e^a \cdot e^b = f(a) \cdot f(b)$

✓

Here's an example of an operation preserving (homomorphism) that is not an isomorphism.

Ex $\varphi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}_2, +)$ by $\varphi(n) = n \bmod 2$.

since:

$$\varphi(a+b) = (a+b) \bmod 2 = a \bmod 2 + b \bmod 2 = \varphi(a) + \varphi(b)$$

(1) φ is a homomorphism

(2) φ is onto: $\mathbb{Z}_2 = \{0, 1\}$, For $0 \in \mathbb{Z}_2$ $0 = \varphi(102) = 102 \bmod 2 = 0$

For $1 \in \mathbb{Z}_2$ $1 = \varphi(3) = 3 \bmod 2 = 1$

(3) φ is not 1-1 b/c $\varphi(102) = \varphi(2)$, yet $102 \neq 2$ in $(\mathbb{Z}, +)$.

Every cyclic group is isomorphic to $(\mathbb{Z}, +)$
infinite

Every finite cyclic group is isomorphic to $\mathbb{Z}_n$ for some n

this follows b/c :

Let $G = \langle a \rangle$ & $|G| = \infty$.

Every element in G is some a^k.

Define $\varphi : G \longrightarrow \mathbb{Z}$ by $\varphi(a^k) = k$.

$$\varphi(\underbrace{a^{k_1} \cdot a^{k_2}}_{a^{k_1+k_2}}) = \varphi(a^{k_1+k_2}) \underset{\text{def'n}}{=} k_1 + k_2 = \varphi(a^{k_1}) + \varphi(a^{k_2})$$

1-1 (onto