MA312 :: Midterm :: Fall 2020

1. Tell me everything you can about $U(18)$. In particular, what are its generators, subgroups, orders of elements, isomorphisms to other groups?

2. Either prove the statement is true or give a counter-example:

   (a) For every $x \in G$ there is some $y \in G$ such that $x = y^2$.

   (b) Let $H = \{z = a + bi \in \mathbb{C}^* \mid a^2 + b^2 = 1\}$. Then $H \leq \mathbb{C}^*$.

   (c) Let $K = \{z = a + bi \in \mathbb{C}^* \mid a^2 + b^2 = 2\}$. Then $K \leq \mathbb{C}^*$.

3. This problem introduces you to three important subgroups of $SL(2, \mathbb{R})$. Prove that T, E, H are all subgroups of $SL(2, \mathbb{R})$.

$$T = \left\{ \begin{pmatrix} 1 & \mathrm{a} \\ 0 & 1 \end{pmatrix} \text{ such that } a \in \mathbb{R} \right\}$$
$$E = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \text{ such that } \theta \in [0, 2\pi] \right\}$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \text{ such that } a \in \mathbb{R} \right\}$$

4. Let G be a group and $a, b \in G$. Use induction to prove that $(bab^{-1})^n = (ba^nb^{-1})$ for every positive integer n. Take care with your induction proof, as I will carefully critique HOW you construct your proof.

5. Assume you can find elements a and b such that $ab = ba$. Prove that a^{-1} and b commute.

6. Prove that the order of ab is the same as the order of ba.

7. Let G be a finite group with more than one element Show that G has an element of prime order.

8. This problem concerns the symmetries of an octagon.

   (a) Construct three (different) subgroups of order 4 in the dihedral group D_8.

   (b) Find the orders of each of the elements of the subgroups you found in part (a).

   (c) Find all subgroups of each subgroup from part (a).

9. Assume $H \leq G$ and let $K = \{x \in G : xax^{-1} \in H \iff a \in H\}$. Prove that $K \leq G$ and that $H \leq K$.

10. Let G be a group and let H be a subgroup of G. For any fixed x in G, define $xHx^{-1} = \{xhx^{-1} | h \in H\}$. Prove the following.

    (a) xHx^{-1} is a subgroup of G.

    (b) If H is cyclic then xHx^{-1} is cyclic.

    (c) If H is abelian then xHx^{-1} is abelian.

    The group xHx^{-1} is called a conjugate of H. (Note that conjugation preserves structure.)

11. Show the group of all rotations of a cube is isomorphic to A_4. Hint: Find four subsets of the solid cube whose permutations correspond to rotations of the cube.

12. Describe and discuss the symmetry (or lack of symmetry) of the physical make-up of the Jamrich building. For the exterior, discuss the patterns of bricks, the colors of the rectangular panels and window sizes. For the interior discuss the patterns in the cinder blocks, the carpet in addition to other design elements. Assume the patterns (bricks, carpets, etc.) extend to infinity.