Thm: A homeomorphism is both an $\underbrace{\text{open map}}_{\text{opens are sent to open sets}}$ & a $\underbrace{\text{closed map}}_{\text{closed sets sent to closed sets}}$.

open map:

let U be open in X &

let $f: X \to Y$ be a homeo. / f & f^{-1} are cts.

show $f(U)$ is open..

f^{-1} is cts $\Rightarrow (f^{-1})^{-1}(U)$ is open if U is open.

b/c $(f^{-1})^{-1}(u) = u$ (this occurs since f is homeo $\Rightarrow f$ is 1-1 so $(f^{-1})^{-1} = f$

$\Rightarrow$ If U is open $f^{-1}(U)$ is open $\Rightarrow f(U) =$ open.

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proof: Let $f =$ homeo: $X \to Y$. Let $g = f^{-1}$. Homeo $\Rightarrow g$ is bijection & cts. (as is f).

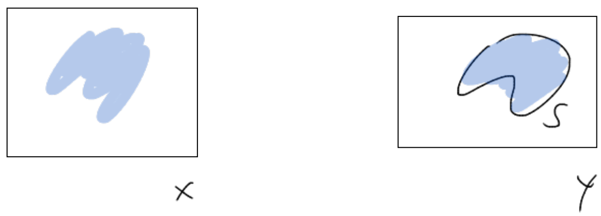
Let U be open in X. $f(U) = V$ & $g(V) = U$. So $g^{-1}(U) = V$. Since g is cts V is open. thus f is an open map.

Let C be closed in X. $f(C) = D$ & $g(D) = C$ by bijection. So $g^{-1}(C) = D$ D is closed since g is cts & C is closed. So f is a closed map

Bijection Property

Let $f: X \longrightarrow Y$ be a bijection. For any set $S \subset Y$

$$f \circ f^{-1}(S) = S$$



properties of functions & complements:

$$f(X - S) = f(X) - f(S)$$

Thm: A homeomorphism $f: X \longrightarrow Y$ is both an open map and a closed map.

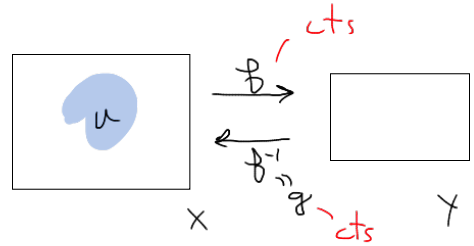
Def'n: Open Map: The image of any open set is open

Closed Map: Image of closed is closed

proof: Let f be a homeo, $f: X \to Y$.

Show f is an open map:

(★) Let $U \subset X$ be open. Show $f(U)$ is open.



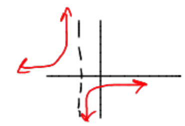
g cts $\Rightarrow g^{-1}(U)$ is open in Y.

$\Rightarrow (f^{-1})^{-1}(U) = f(U)$ b/c f is a bijection

[Let $C \subset X$ be closed. $X - C$ is open, above (★) $\Rightarrow f(X-C)$ is open so $f(X) - f(C)$ is open, bijection $\Rightarrow f(X) = Y$ so $Y - f(C)$ is open $\Rightarrow f(C)$ is closed]

Recall: $f: X \to Y$, $A \subset X$

if f cts then $f(\overline{A}) \subset \overline{f(A)}$



Ex: $A = \mathbb{R}$, $X = (\mathbb{R}^2, \text{std})$, $\overline{A} = A$ so $f(\overline{A}) = f(A) =$  $= \overline{f(A)}$

 $= f(A)$

 $= \overline{f(A)}$

$= f(\overline{A})$

Ex $(0,10) = A$, $[0,10] = \overline{A}$, $f(x) = \frac{1}{x+1}$

Thm: If f is a homeo $f: A \to B$ then $f(\overline{A}) = \overline{f(A)}$

proof f homeo $\Rightarrow f$ cts $\Rightarrow f(\overline{A}) \subset \overline{f(A)}$

So now show $\overline{f(A)} \subset f(\overline{A})$

Since f^{-1} cts, remark above $\Rightarrow f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$

Bijection $\Rightarrow B = f(A)$

$\Rightarrow f^{-1}(\overline{f(A)}) \subset \overline{f^{-1}(f(A))} = \overline{A}$

$f \circ (f^{-1}(\overline{f(A)})) \subset f(\overline{A})$

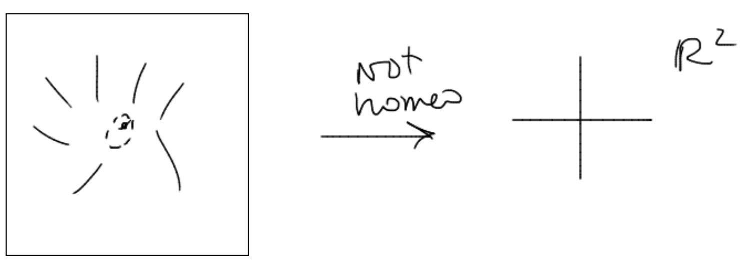
$\overline{f(A)} \subset f(\overline{A})$

Thm: If $f: X \to Y$ is a homeo then $f(\mathring{X}) = \mathrm{Int}(f(X)) = \mathrm{Int}(Y)$

f maps the interior to the interior.

Thm: For a homeo $f: X \to Y$ $\quad f(\partial X) = \partial(f(X))$

f preserves the boundary.



Friday – Week 10

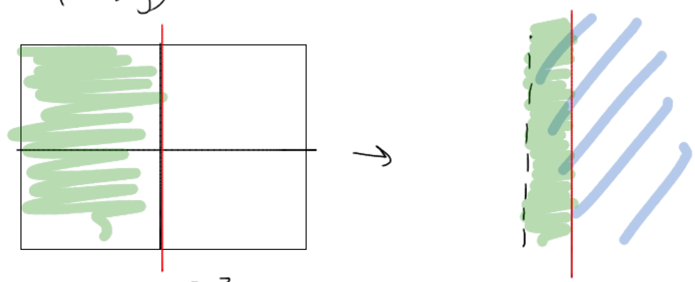
Ex 1

$\mathbb{R}^2$ = plane

$f(x,y) = (e^x, y)$

$\mathbb{R}^2 \simeq \mathbb{H}$

$\{(x,y) \in \mathbb{R}^2 \mid x > 0\}$

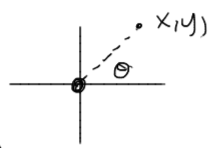


y-axis $= \{(0,y) \mid y \in \mathbb{R}\}$

$f(\text{y-axis}) = \{(1,y)$

Ex 2

$\mathbb{R}^2 \simeq \mathring{D} = \{$open unit disk$\}$



$$(x,y) \longleftrightarrow (\theta, r) \xrightarrow{\quad f \quad} \left(\theta, \frac{r}{1+r}\right)$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{x^2+y^2}$

f is 1-1 / onto