EXAMPLE 4.13. Let $[0, 2\pi)$ and S^1 have the standard topology as subspaces of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. We denote each point in S^1 by p_θ, where p_θ represents the point on S^1 at angle $\theta \in \mathbb{R}$, measured counterclockwise from the positive x-axis. Define $f : [0, 2\pi) \to S^1$ by $f(\theta) = p_\theta$, as illustrated in Figure 4.12. It is clear that f is a bijection.

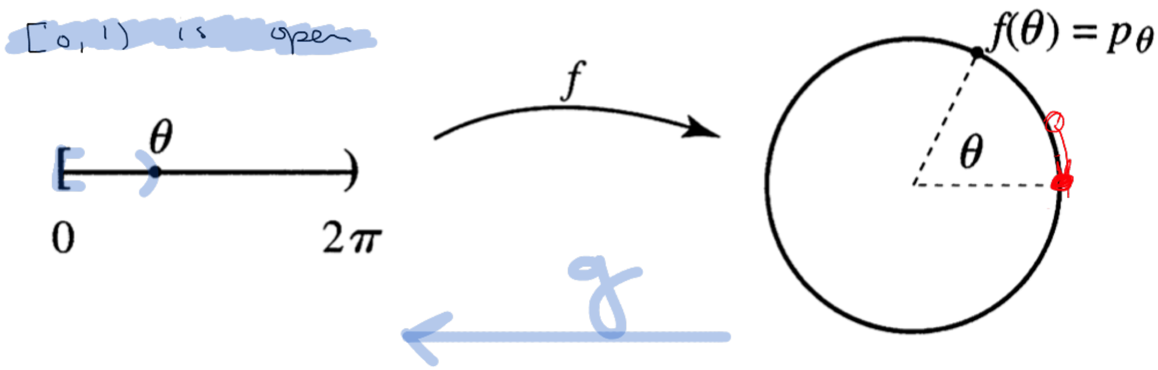


FIGURE 4.12: A contin~~uous bijection from~~ $[0, 2\pi)$ onto S^1 that is not a homeomorphism.

4.2

Wed. Week 10

f is cts if: Preimage of any open set in Y is open in X; $f: X \longrightarrow Y$

Homeomorphisms:

A map $f: X \longrightarrow Y$ is a homeo if:

1. f is a bijection (1-1 and onto)
2. f is cts
3. f^{-1} is cts

Ex. $f(x) = 3x+1$ is homeo from $\mathbb{R} \to \mathbb{R}$.

$y = 3x+1$

$\frac{y-1}{3} = x$ so $f^{-1}(x) = \frac{x-1}{3}$ is also cts

Ex: $f(x) = e^x$, $f: \mathbb{R} \longrightarrow \mathbb{R}^+$ is a homeo: $(-\infty, \infty) \overset{\text{homeo}}{\cong} (0, \infty)$

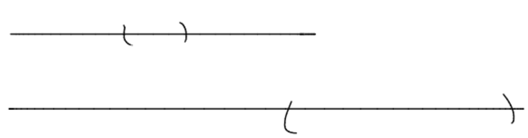
$f^{-1}(x) = \ln(x)$

Ex. $(2,4) \cong (e^2, e^4)$

A

Ex $(0,1) \cong (a,b)$ where $a < b \in \mathbb{R}$

Find f such that $f((0,1)) = (a,b)$ & f, f^{-1} to be cts



send $f(0) = a$ $\quad f(x) = x + a$

stretch $(0,1) \longrightarrow (a,b)$

$g(x) = (b-a)x$

compose these two:

$h(x)$ $f \circ g(x) = f((b-a)x)$

$= (b-a)x + a$

$h(0) = a$

$h(1) = (b-a)\cdot 1 + a = b$.

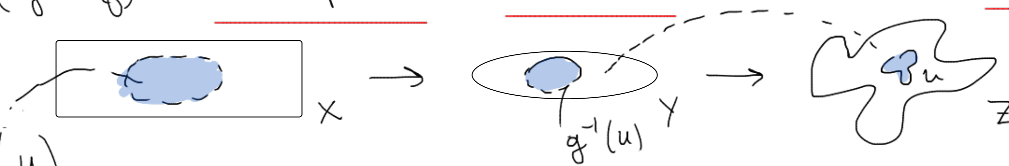
h is cts b/c the composition of cts fcns is cts

Theorem: If f, g are cts: $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$

then $(g \circ f)(x)$ is cts.

Let U be an open set in Z. Show preimage of U under $(g \circ f)$ is open in X.

Fact:



$$f^{-1} \circ g^{-1}(u)$$

$$\shortparallel$$

$$f^{-1}(g^{-1}(u)) = f^{-1}(\{y \in Y \mid g(y) \in U\})$$

$$= \{x \in X \mid f(x) = y \in Y \text{ and } g(y) \in U\} = \{x \in X \mid g(f(x)) \in U$$

$$= (g \circ f)^{-1}(u)$$

Contravariance

$$\boxed{(g \circ f)^{-1}(u) = (f^{-1} \circ g^{-1})(u)}$$

$$(g \circ f)^{-1}(u) \underset{\text{Fact}}{=} (f^{-1} \circ g^{-1})(u)$$

$$= f^{-1}(\underbrace{g^{-1}(u)}_{\substack{\text{open in } Y \\ \text{b/c } g \text{ is cts}}}) = \overbrace{f^{-1}(\text{some open set in } Y)}^{\text{open in } X \text{ b/c } f \text{ is cts}}$$

So far : $(0,1) \cong (a,b)$

<u>Ex</u>. $(a,b) \simeq (0,\infty)$

perhaps / arbitrarily small

we'll use the Ex below.

$h : (a,b) \longrightarrow (0,1)$ Ex A

f^{-1} below : $(0,1) \longrightarrow (0,\infty)$

$(f^{-1} \circ h)$ is a homeo b/w (a,b) & $(0,\infty)$

<u>Ex</u> $(0,\infty) \simeq (0,1)$

$$f(x) = \frac{1}{1+x}$$