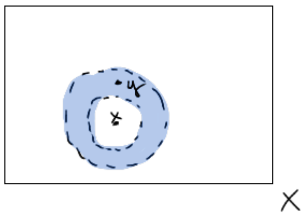
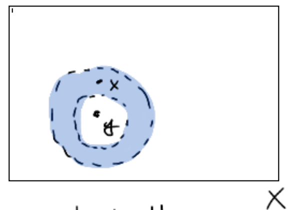
5.13 (X,d) metric space $d: X \times X \to \mathbb{R}$ is cts if top on $X \times X$ is product

Let $\mathcal{U} \subset \mathbb{R}$ be open. $d^{-1}(\mathcal{U}) = \{(x,y) \mid x \in X, y \in X, d(x,y) \in \mathcal{U}\}$.

$\mathcal{U} = (a,b)$ so $d(x,y) \in \mathcal{U} \Rightarrow d(x,y) > a$ and $d(x,y) < b$.



$y \in$ b-ball about x open in metric top



$x \in$ b-ball about y open in metric top

$d^{-1}(\mathcal{U}) \subset$ (b-ball about y) $\times$ (b-ball about x)

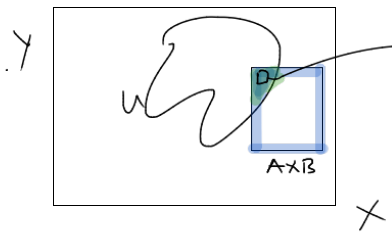
3.20:

show $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$

Let $(p,q) \in \overline{A \times B}$	show $p \in \bar{A}$ & $q \in \bar{B}$.

One Strategy

Assumption $\Rightarrow$ U arbitrary open nbhd of (p,q) must intersect $A \times B$.



$V_1 \times V_2$, $p \in V_1$ open in X, $q \in V_2$ open in Y $\Big\}$ $V_1 \cap A \neq \phi$, $V_2 \cap B \neq \phi$

$\underset{\text{error}}{\Rightarrow}$ $p \in \bar{A}$, $q \in \bar{B}$ both V_1 & V_2 are arbitrary

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Let V_1 be open nbhd of p in X, V_2 open nbhd of q in Y

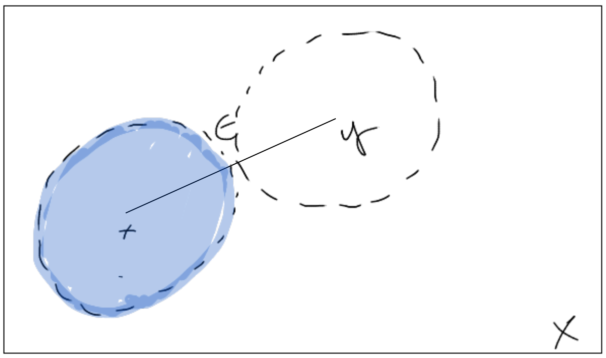
show $V_1 \cap A =\phi$ and $V_2 \cap B \neq \phi$, Proceed by contradiction

If not $V_1 \times V_2$ open, contain (p,q), and disjoint from $A \times B$.

Metric Spaces

Thm: Any metric space is Hausdorff

Let (X,d) be a metric space. Let $p,q \in X$. Find open sets about p,q resp. that are disjoint.



$\boxed{\text{Let } \epsilon = d(x,y)}$. Now just use $\frac{\epsilon}{2}$ balls about x & y

$U = B(x, \frac{\epsilon}{2}) = \{w \in X \mid d(x,w) < \frac{\epsilon}{2}\}$

$V = B(y, \frac{\epsilon}{2}) = \{w \in X \mid d(y,w) < \frac{\epsilon}{2}\}$

Claim $U \cap V = \phi$. Assume not. Let $z \in U \cap V$.

1. $d(x,z) < \frac{\epsilon}{2}$
2. $d(y,z) < \frac{\epsilon}{2}$

symm $\quad = d(z,y)$

Δ-Ineq:

$$\boxed{d(x,y) \leq d(x,z) + d(z,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon}$$

$\times$

Connectedness

Def'n: Connected

1. Not disconnected
2. Def'n Disconnected: there exists a separation
3. Def'n Separation: A separation of a space X is two disjoint open sets whose union is X.

EX $(\mathbb{R}, std)$ is connected

$(-\infty, 1) \cup (0, \infty) = \mathbb{R}$

(not disjoint

$(-\infty, 1) \cup (1, \infty) \neq \mathbb{R}$

$(-\infty, 0) \cup \{0\} \cup (0, \infty) = \mathbb{R}$

disjoint (not open

Ex. $(\mathbb{R}, LLT)$ is disconnected

$(-\infty, 1) \cup [1, \infty)$

Ex. Is $[1, 10)$ in $(\mathbb{R}, LLT)$

$[1,10) = [1,4) \cup [4,10)$

$[1,4) \to [1,2) \cup [2,4)$

$\to$ ... continue

Totally Disconnected

[The only connected components in $(\mathbb{R}, LLT)$ are singletons]

$\{5\} = U \cup V$

$U =$ open

$V =$ open

$U \cap V = \emptyset$

How to tell when subsets are connected?

Idea:

Let $X = \mathbb{R}$ w/ std top.

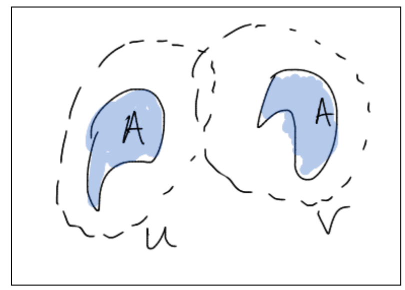
$A \cap (3,5) = \phi$

$A = [0,1) \cup \{5\}$

$U = (-1,5)$, $V = (3,6)$, $A \cap U = [0,1)$, $A \cap V = \{5\}$, $A \cap U \cap V = \phi$

Thm: The subset A is disconnected in X if $\exists$ open U, V in X s.t.

1. $A \subset U \cup V$
2. $A \cap U \neq \phi$
3. $A \cap V \neq \phi$
4. $A \cap (U \cap V) = \phi$

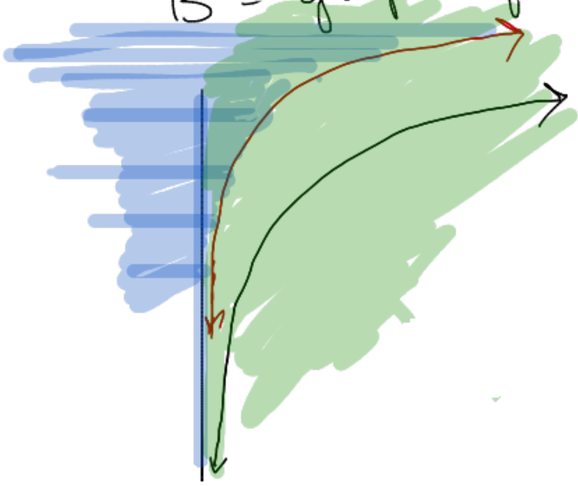


X

A is contained in two open sets & A intersects both, but not their intersection

Ex A = y-axis in $\mathbb{R}$ = $\{(0,y) \mid y \in \mathbb{R}\}$

B = graph of $y = \ln(x)$ = $\{(x, \ln(x)) \mid x \in \mathbb{R}\}$



Let $C = A \cup B$.

Is C connected?

Hint: $y = \ln(x) + 1$

U = set of points to left of graph of

V = set of points to right of y-axis.