Theory of Connectedness

1. <u>thm</u>: cts image of connected is connected.

   Suppose: X conn, f cts. If U, V form separation of $f(X)$ in Y
   $f: X \to Y$

   $f^{-1}(U), f^{-1}(V)$ are open disjoint & $f^{-1}(U) \cup f^{-1}(V) \supset X$.

   $\Rightarrow f^{-1}(U)$ & $f^{-1}(V)$ form separation of X.

2. Homeomorphisms preserve connectedness.

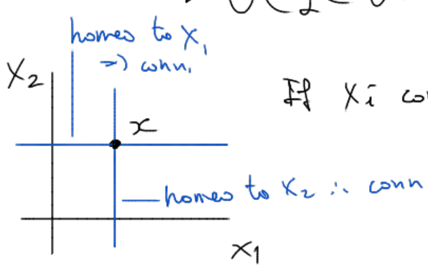
3. Product of Connected is Connected

   <u>lemma</u>:  C_α connected $\bigcap_\alpha C_\alpha \neq \phi \Rightarrow \bigcup C_\alpha$ connected.

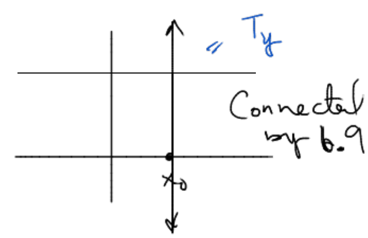
   <u>proof</u>: Let U, V form separation of $\bigcup C_\alpha$. Let $x \in \bigcap C_\alpha$ be arbitrary. Sep. $\Rightarrow x \in U$ or V, not both, say U.

   $\Rightarrow \bigcap C_\alpha \subset U \Rightarrow$ (Connected Containment lemma) $\Rightarrow C_\alpha \subset U \ \forall \alpha$

   $\Rightarrow \bigcup C_\alpha \subset U$ ⊗ sep.



If X_i connected $X_1 \times X_2 \bigcup_{y \in X_2} T_\alpha$



Adding Limit Points preserve Conn.

$C \subset A \subset \overline{C}$.
conn.

If U, V sep. A, (6.7) $\Rightarrow C \subset U$

Yet $A \cap V = \phi$.

For $x \in A \cap V$, $x \in A \subset \overline{C}$, every nbhd x intersects C.

But $x \in V$ w/ $V \cap C \neq \phi$.

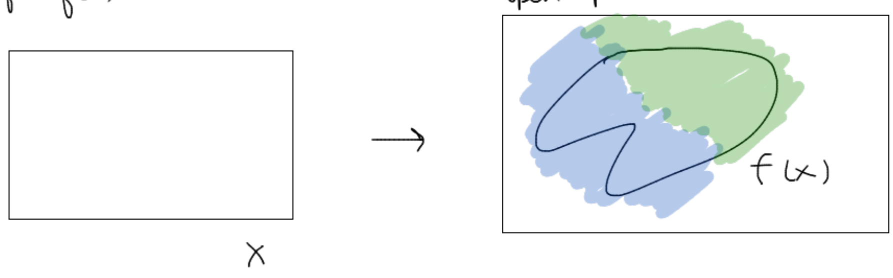
Monday – Week 12

Theory of Connectedness:

<u>Thm</u>: The continuous image of a connected set is connected.

$f: X \to Y$, $f(X)$ is connected

<u>proof</u>: If $f(X)$ is disconnected, let $\underset{\text{open}}{U}, \underset{\text{open}}{V}$ be a separation of $f(X)$ in Y.



$f^{-1}(U)$ and $f^{-1}(V)$ are open b/c f is cts,

are disjoint b/c $U \cap V = \phi$

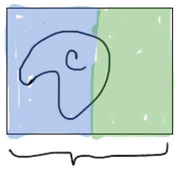
$\left(x \in f^{-1}(U) \cap f^{-1}(V) \Rightarrow f(x) \in U \text{ and } f(x) \in V \right)$

& $f^{-1}(U) \cup f^{-1}(V) \supset X$ b/c $f(X) \subset U \cup V$.

<u>Cor</u> Homeos preserve connectedness

Lemma: If "something connected", say C, contained in disconnected set D then C is in exactly one of the open sets disconnecting D

Idea:



D

If so, the separation of D descends to a separation of C.

More formally, If connected set $C \subset D$ & U, V are separation of D

$$\Rightarrow C \subset U \text{ or } C \subset V \text{ but \underline{not} both}$$

Thm: "Adding a limit point to a space preserves connectedness."

If C is connected & $C \subset A \subset \overline{C}$

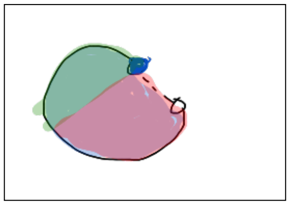
then A is connected.

Recall: $\overline{C} = C \cup C'$ so this $\uparrow$ becomes $C \subset A \subset C \cup C'$

$\Rightarrow A = C \cup$ some limit pts of C.

$\boxed{A \subset \underset{\text{open}}{U} \cup \underset{\text{open}}{V},\ A \cap V = \phi,\ A \cap U = \phi \quad U \cap V = \phi}$

Proof:



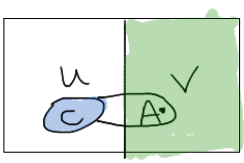
If $\boxed{U, V \text{ form a separation of } A}$ then apply previous Lemma: (let $A = D$)

$\Rightarrow$ assume $C \subset U$.

$C \cap U \neq \phi$ & $C \cap V = \phi$.

But sep $\Rightarrow A \cap V \neq \emptyset$

so let $x \in A \cap V$.



$x \in A \subset \overline{C}$. So every open nbhd of x intersects C.

But V is an open nbhd of x disjoint from C. $\otimes$

$\Rightarrow$ No such separation exists.

$\Rightarrow A$ connected.

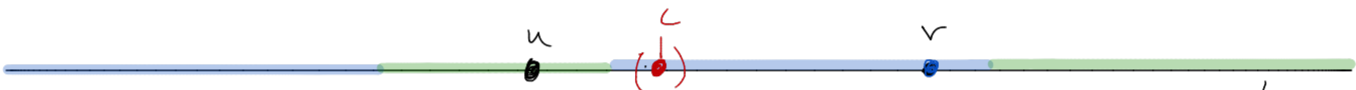
Thm: $(\mathbb{R}, \text{std})$ is connected

Recall: $(\mathbb{R}, LLT)$ is not connected. $(-\infty,1)\cup[1,\infty)$ is a separation

Assumption:

1. Every subset of reals that is "bounded above" has a Least Upper Bound.
2. If $x<y$ $x,y\in\mathbb{R}$ $\exists$ z s.t. $x<z<y$.

proof: Assume (by way of contradiction) (ABWOC) that U, V form a separation of $\mathbb{R}$.



Let $u\in U$, $v\in V$, $u<v$.

Define: $U' = U\cap[u,v]$

$V' = V\cap[u,v]$

v is an upper bound on U', so let c be the Least Up. Bound of U'.

If $c\in V'$ then, note V' is open in $[u,v]$ so $\exists$ an open-interval about c in V' so $c\in(d,e)\subset V'$. So $(d,c]\subset V'$.

Thus d is an Upper Bound on U' so c is not the L.U.B. on U' $\otimes$.

If $c\in U'$, open in $[u,v]$ so $(x,y)\ni c$ & $(x,y)\subset U'$. So $[c,y)$ since $y\in U'$ and $y>c$ c is not even an upper bound on U'. $\otimes$. So $c\notin U'$, $c\notin V'$.

$U'\cup V' = [u,v] \Rightarrow$ No separation exists.