Wed - Week 12

Exercises: 6.1, 6.5, 6.17, 6.20, 6.27

Due: Wednesday

Continue Theory of Connectedness

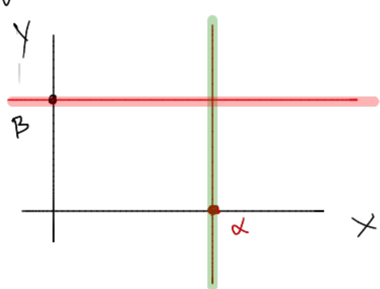
We know so far: $\mathbb{R}$ is connected (in std. top) & anything homeomorphic to it

Goal: $\mathbb{R}^2$ is connected. $\mathbb{R}^n$

Main Tool: Connected Containment Lemma:

If $\{C_\alpha\}$ are connected sets & their common intersection is non-empty $\bigcap_\alpha C_\alpha \neq \phi$ then $\bigcup_\alpha C_\alpha$ is connected.

Thm. If X & Y are connected the $X \times Y$ is connected



Let $C_\alpha = \{\alpha\} \times Y \cup X \times \{\beta\}$

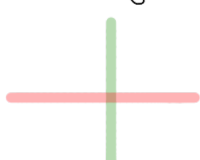
each is connected b/c

$\{\alpha\} \times Y \underset{\text{homeo}}{\cong} Y$

$X \times \{\beta\} \cong X$

By letting β vary we get an ∞ # of

whose union is all of $X \times Y$.



& their common intersection is $\{\alpha\} \times Y$

$\Rightarrow$ So their union is connected.

Fact: S^2 is connected. $S^2 =$ 

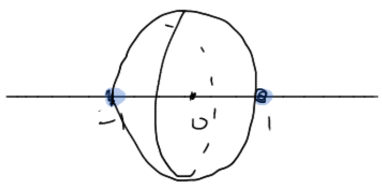
$S^2 - \{x\}$ for some $x \in S^2$, is homeomorphic to $\mathbb{R}^2$, & thus
$S^2 - \{x\}$ is connected

In std top on $S^2 \subset \mathbb{R}^3$ x is a limit pt.

(*) Adding L.P.'s preserves connectedness

This generalizes to all S^n, $n \geq 1$

$S^3 =$? ? ?



S^0

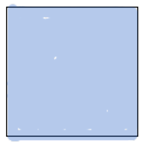
$\mathbb{R}$

Fact $I \times I$ is connected $\quad I = [0,1]$

$$= \{0\} \cup (0,1) \cup \{1\}$$

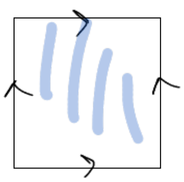
L.P. $\quad (0,1) \cong \mathbb{R}$ conn $\quad$ L.P

why?

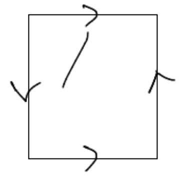


Product of Connected is Connected

Fact



or



each is connected

quotient maps are <u>continuous</u>.

And <u>cts</u> <u>fcns</u> preserve connectedness

Tool for distinguishing spaces via connectedness

Cutset: Set of points whose complement is disconnected.

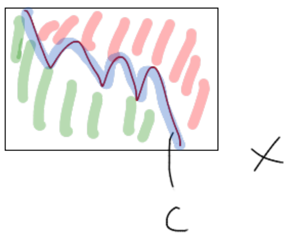
"cut points separate the space".

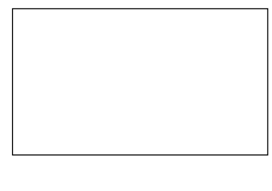
~~Fact~~ Thm: Homeomorphisms preserve cutsets.

proof: Let C be a cutset of X. $X - C$ is disconnected.

Say $X - C = U \cup V$ where each U, V is connected

Let f be a homeo $f: X \to Y$.





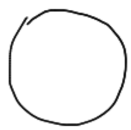
If $Y - f(C)$ is not disconnected (If $f(C)$ is not a cutset)

then $Y - f(C)$ is connected. f^{-1} is cts & cts image of connected is connected so

$f^{-1}(Y - f(C))$ must be connected but ...

this is exactly $Y - C$. $\times$.

Fact S^1 not homeomorphic to $\mathbb{R}$

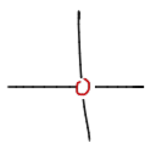


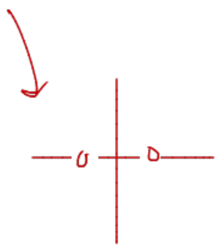
NO x is a cutpoint

every x is a cutpoint

Fact $S^2 \not\cong \mathbb{R}^2$ b/c

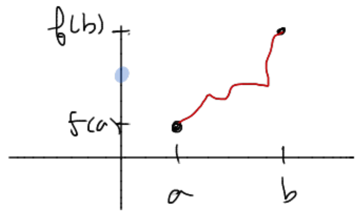
Remove two pts



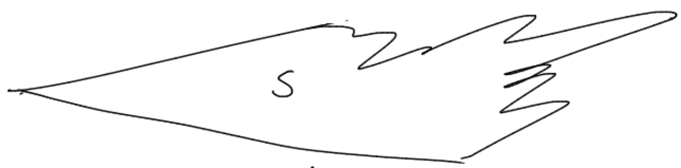


Intermediate Value Theorem

(In Calculus: If $f:[a,b] \to \mathbb{R}$ is continuous and $f(a) < y < f(b)$
$\Rightarrow \exists\ c \in [a,b]$ s.t. $f(c) = y$



Turns out this true for any connected domain



Surface of Lake Sup

$f: S \to \mathbb{R}$

$f(x) =$ depth of water below x

max depth $= 1280'$,

$\exists\ a$ s.t. $f(a) = 1280'$

$\exists\ b$ s.t. $f(b) = 0$

I.V.T says