Friday - Week 13

compactness

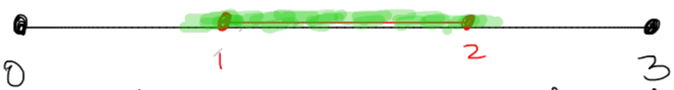
closed Hausdorff

Thm: If C is closed & $C \subset D$, w/ D compact, then C is compact.

Ex: $(\mathbb{R}, \text{std})$, Let $D = [0,3]$ is compact. (We showed in class that $[-1,1]$ is compact & $[0,3] \cong [-1,1]$ & homeos preserve compactness)

Let $C = [1,2]$. Clearly $[1,2]$ is cpct.

Given: $\{U_\alpha\}$ is open cover of $[1,2]$.



what can we add to the $\{U_\alpha\}$ collection to get an open cover of $[0,3]$

$V_\beta = \mathbb{R} - C \bigcup \{U_\alpha\}$ is open cover of $[0,3]$, a compact set. We throw out all but finitely many $\{V_\beta\}$ to get a finite cover of $[0,3]$

only the U_α cover $[1,2]$

$\mathbb{R} - C \bigcup \{U_i\}_{i=1}^N$ open cover of $[0,3]$. Show $\{U_\alpha\}$ have a finite subcover of $[1,2]$

Observe: $\{U_i\}_{i=1}^N$ is open, finite cover of $[1,2]$, $\therefore$ $[1,2]$ is compact.

proof: (of main Thm). Replace $[0,3]$ by some arb. compact set D, $[1,2]$ by some arb. closed set $C \subset D$.

Ex $(\mathbb{R}, \text{F.C.T.})$ (non-Hausdorff). $D = \mathbb{R} - \{1,2,3\}$. Claim: D is compact. Choose our compact set

$C = \{4,5,6\}$ is closed. Also, C is compact.

Note: C is covered by at most 3 open sets.

Intersections:

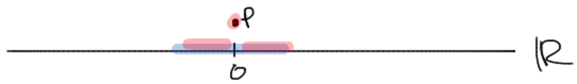
open sets: $U_\alpha \Rightarrow \bigcap_{i=1}^{N} U_i$ is open — Finite

closed sets: $C_\alpha \Rightarrow \bigcap_{\alpha \in A} C_\alpha$ is closed — arbitrary

compact sets: $K_\alpha \Rightarrow$ If Hausdorff $\bigcap_{\alpha \in A} K_\alpha$

Ex. Double Origin Topology on $\mathbb{R}$.

open sets:
1. (a,b) if $0 \notin (a,b)$
2. $(a,0) \cup \{p\} \cup (0,b)$ $\quad a<0,\ b>0$



$(\mathbb{R}, D.O.T.)$ not Hausdorff.

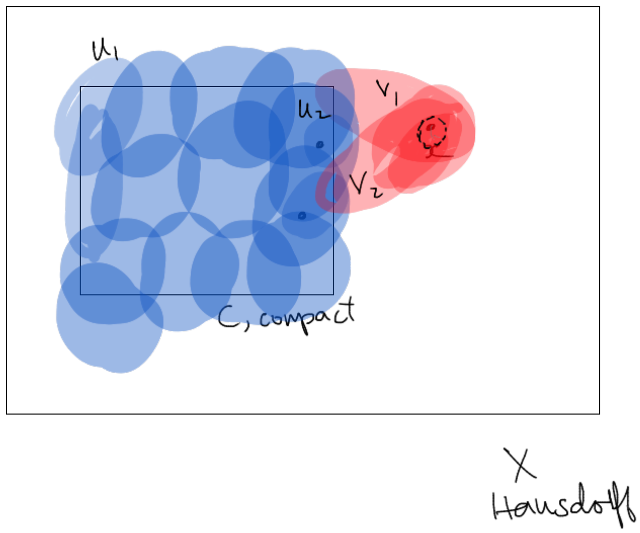
Claim: $[-1,1]$ is compact & $[-1,0) \cup \{p\} (0,1]$ is compact

But their intersection is: $[-1,0) \cup (0,1]$ isn't compact

$\{[-1,-\frac{1}{n}) \cup (\frac{1}{n},1]\}_{n=1}^{\infty}$ yet this covers.

No finite sub-cover.

Thm: Compact sets in Hausdorff spaces are closed.



Show C is closed. Show $X-C$ open

Let $x \in X-C$. Apply Hausdorff prop. to every $y \in C$. get an open cover $\{U_\alpha\}$ of C.

So $\{U_i\}_{i=1}^N$ covers C.

and $x \in \bigcap_{i=1}^N V_i$

w/ V_i open ... so is open & is disjoint from C. (Hausdorff)

$\Rightarrow X-C$ is open

C closed