Monday

1. compactness
    a. def'n
    b. R^n is not compact
    c. finite spaces are
    d. def'n: A compact in X
        i. every cover of A by open sets in X has a finite subcover
    e. ex: A = {1/n} U {0} is compact
    f. (0,1] is not compact in R
    g. thm: cts image of compact is compact
    h. THM: unions and intersections
        i. finite unions of compact are compact
        ii. if Hausdorff, intersection of compacts are compact
            1. extra point line
    i. CLOSED and COMPACT are related
        i. THM: Closed inside compact is compact
            1. proof: C in D. Add X-C to an open cover of D.
            2. example: R^n
        ii. Not the same:
            1. (R, FCT) every set is compact. not every set is closed.
        iii. THM: Hausdoff + Closed = Compact
            1. A closed, let a in A and x not in A. Hausdorff gives disjoint nbhds, a in V_a, x in U_a. Do this for all points a, but keeping x fixed. The union of all the V_a is open cover of A. Thus there is a finite subcover indexed by I. Then union over I of V_a is open, and intersection over I of U_a is also open containing x. These are disjoint, so U_a misses A entirely. x is arbitrary, so the complement is open
        iv. Product of compact is compact.

Compactness – Wed– week 13

Def'n: A set X is compact if any open cover has a finite sub-cover.

(Recall: A basis requires $\forall x \in X$ $\exists$ basis elt. containing it (the basis elts "cover" X).

open-cover: collection of open sets that "cover", i.e., $\forall x \in X$ $\exists$ open set $U \ni x$

sub-cover: subset of elements from the given cover

finite sub-cover: only a finite # of sets from the cover.

Ex: $X = \{a, b, c\}$, $T = \{\{a,b\}, \{b,c\}, \{b\}, \phi, X\}$

An open cover of X is $C_\alpha = \{\{a,b\}, \{b,c\}\}$. X is compact.

Fact Finite Topological Spaces are compact.

Ex $(\mathbb{R}, std)$ is non-compact.

Even though $\{(-\infty, 0), (-1,1), (0,\infty)\}$ is a finite cover of $\mathbb{R}$.

Here's an infinite cover of $\mathbb{R}$ with no finite subcover

$C_\alpha = \{(0,2), (1,3), (2,4), (3,5), \ldots$
$\ldots (-1,1), (-2,0), (-3,-1), (-4,-2) \ldots\}$

1. C covers $\mathbb{R}$
2. removing sets to make it finite would force C to miss some points.

Ex $[-1,1]$ is compact in $(\mathbb{R}, std.)$

$C = \{(-1, 1-\frac{1}{n}) \mid n \in \mathbb{N}\}$, $U = (-2, -.75)$, $V = (.9, 2)$

$C_\alpha = C \cup U \cup V$ is an infinite open cover of $[-1,1]$.

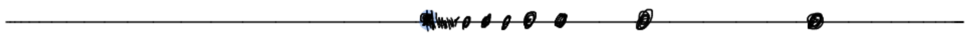
$(-2,2)$ does cover $[-1,1]$, but this irrelevant b/c $(-2,2)$ is not a member of the given cover.

Let $N = 20$ then $C_N = \{(-1, 1-\frac{1}{n}) \mid n \leq N\}$ "20"

$(-1,0), \ldots (-1, 1-\frac{1}{20})$ is a sub cover.
$1-\frac{1}{20} = .95$

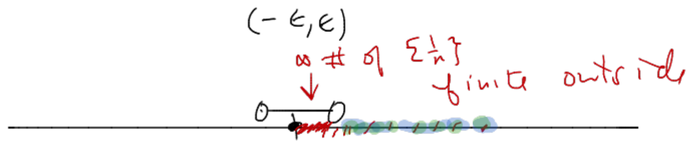
Now $C_N \cup U \cup V$ is a finite subcover.

EX $A = \{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$ is in $(\mathbb{R}, std)$



Let C be an arbitrary open cover of A.
Show C has a finite sub-cover.

zoom



$0 \in C$ so $\exists$ open set in C, U, s.t. $x \in U$. (there must be some open-interval containing 0 that is a member of C.

Since $\{\frac{1}{n}\} \longrightarrow 0$ $\exists$ N s.t. $\frac{1}{n} \in U$ if $n \geq N$.

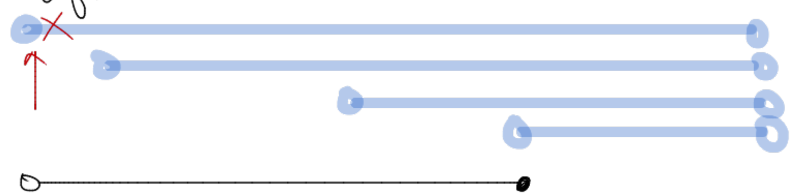
$\frac{1}{n} < \epsilon$ or $\frac{1}{\epsilon} < n$

Your subcover is U together with the finite # of open sets that cover $\{\frac{1}{n}\}$ for $n < N$.

Ex. $(0,1]$ is not compact in $(\mathbb{R}, \text{std})$. No finite sub-cover of $\{(\frac{1}{n}, 2) \mid n \in \mathbb{N}\}$.

Any finite sub-cover would have a max N s.t. $(\frac{1}{N}, 2)$ is in the sub-cover and nothing to the left. this would not cover $\frac{1}{N+1} \in (0,1]$

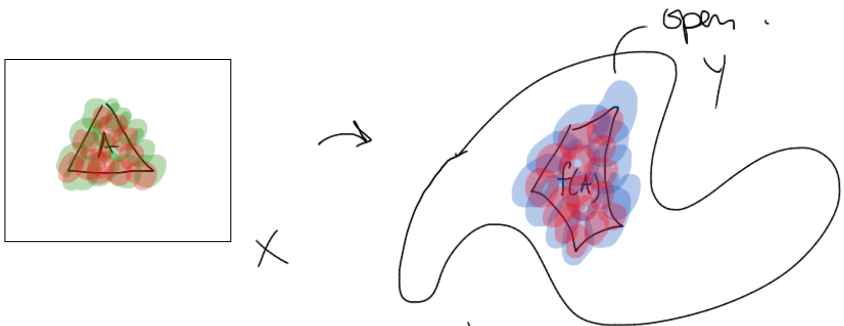
$0 < \frac{1}{N}$



Thm: The continuous image of a compact set is compact

Proof: Let $f: X \to Y$ be continuous & let $A \subset X$ be a compact set. We show $f(A)$ is compact in Y.

Let C_α be an open cover of $f(A)$



$f^{-1}(C_\alpha)$ is open in X.

C_α covers $f(A)$

$f^{-1}(C_\alpha)$ open cover of A.

$\Downarrow$ compactness

$\exists$ finite open subcover of $f^{-1}(C_\alpha)$ of A

Now $f\left(\text{finite subcover of } A\right) =$ finite subcover of C_α that covers $f(A)$