Monday — Week 14

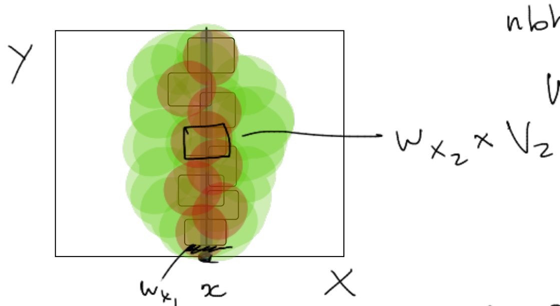
Last HW due: Dec. 9 (Final)

Course Summary: Dec. 9

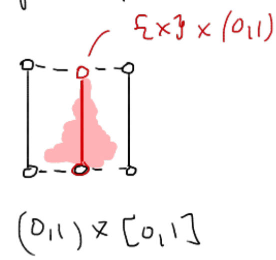
Last Time: Compact Sets are a lot like closed sets. In fact, when the space is Hausdorff, compact $\Rightarrow$ closed.

To show Products of Compact sets are compact we need:

The Tube Lemma: Given $X \times Y$ w/ Y compact any nbhd of $\{x\} \times Y$ contains a tube $W_x \times Y$



why compactness?



Proof: Assume $\mathcal{U}$ is open cover in $X \times Y$, nbhd of $\{x\} \times Y$. This set is homeo to Y, so it's compact b/c homeos preserve cpct. Thus a finite # of open sets from $\mathcal{U}$ cover $\{x\} \times Y$, call them $W_{x_i} \times V_i$. So $\bigcup_{i=1}^{N} V_i$ covers Y

$Z = \bigcap_{i=1}^{N} W_{x_i} \ni x$ & is open. Now $Z \times Y$ is a tube containing $\{x\} \times Y$.

Thm: Let X, Y be compact. Then $X \times Y$ is compact

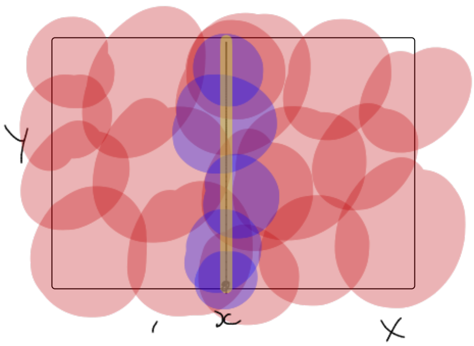
proof:

Let $\mathcal{O}$ be an open cover of $X \times Y$.

$\forall x \in X$, $\{x\} \times Y$ is covered by $\mathcal{O}$ & is compact, therefore

let $\mathcal{O}_x$ be the finite subcover of $\mathcal{O}$ that covers $\{x\} \times Y$.

$\mathcal{O}_x$ contains a tube by Tube Lemma,



$\{x\} \times Y \subset W_x \times Y$

$\forall x \in X$ $\{W_x\}_{x \in X}$ covers X ∴

'inf'

cptness of $X \Rightarrow \{W_{x_i}\}_{i=1}^{N}$ covers X.

$\{\mathcal{O}_{x_i}\}_{i=1}^{N}$ finite subset of $\mathcal{O}$

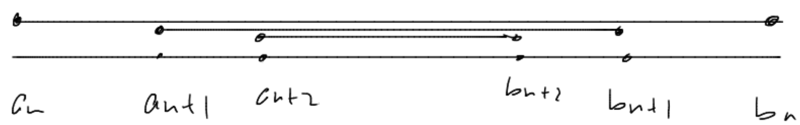
∴ finite subcover

Compactness In Metric Spaces

[Hausdorff + Compact = Closed & bounded]

Nested Intervals Lemma:

For $[a_n, b_n] \subset \mathbb{R}$ w/ $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$



Ex

$[-\frac{1}{n}, \frac{1}{n}]$ $n \in \mathbb{N}$

$[-\frac{1}{3}, \frac{1}{3}] \subset [-\frac{1}{2}, \frac{1}{2}]$

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$$

closed is neccessary!

proof: If A = least upper bound of $\{a_n\}$
B = greatest lower bound of $\{b_n\}$. $\Longrightarrow \bigcap_{n=1}^{\infty} [a_n, b_n] = [A, B]$

key ingredient to:

Thm: Every closed & bounded interval is compact in std. top. on $\mathbb{R}$.