Monday - Week 2

Homework # 1 (Due Wed.)

$\begin{cases} X - A_i & \text{finite} \checkmark \\ \phi \end{cases}$

Discrete Top vs. Finite Complement Top

Ex 1

$X = \cdot$

$T = \{X, \phi\}$ (= discrete top = finite compl. top)

Ex 2

$X = \{A, B, C\}$

$T_1 = \{\{A,B,C\}, \{A,B\}, \{A,C\}, \{B,C\}, \{A\}, \{B\}, \{C\}, \phi\}$ discrete top on X,

Let's build the Finite Compl. Top on X

$T_2 = \{\{C\}, \{A\}, \{B\}, \ldots \} = T_1$

This works whenever $|X| < \infty$. (finite)

Ex. ($\mathbb{Z}$, discrete top :))

Ex ($\mathbb{Z}$, finite compl. top) $\mathbb{Z} - \{1,2,3\} =$ open subset

Discrete Top $\iff$ $\{x\} \in T$

(all singletons are open)

Iff proofs require 2 sub-proofs:

$\Rightarrow$ Assume X has the discrete top, prove $\{x\} \in T$

Every subset of X is open by assumption.

$\Leftarrow$ Assume $\{x\} \in T$ prove X has D.T. (every subset is open)

Let S be an arbitrary subset of X.

Hint: $\mathbb{R} = \bigcup_{r \in \mathbb{R}} \{r\}$ (every set is a union of its elements)

Show the Finite Complement Topology on $\mathbb{R}$ is actually a topology.

Let's recall

$T = \{ \phi$, any set whose complement is finite$\}$
(or unions)

1. $\phi \in T$ by def. $\mathbb{R} \in T$ b/c $\mathbb{R} - \mathbb{R} = \phi \in T$

3. Finite Intersections of members of T are themselves members of T.

2. Inf. unions ..

Let $A_i \in T$. Show $\underbrace{(\mathbb{R} - A_1) \cap (\mathbb{R} - A_2) \cap \ldots \cap (\mathbb{R} - A_n) \cap \ldots}$

$\bigcap_{i=1}^{n} A_i$ open

$\boxed{\bigcup_{i=1}^{\infty} A_i \text{ open}}$

$= \bigcap_{i=1}^{\infty} (\mathbb{R} - A_i) \longrightarrow$ Infinite intersection of Finite is finite!

De Morgan

$= \underbrace{\mathbb{R} - \bigcup_{i=1}^{\infty} A_i}_{\text{The complement of } \bigcup_{i=1}^{\infty} A_i}$

So $\bigcup_{i=1}^{\infty} A_i \in T$!

3. Show $\bigcap_{i=1}^{n} A_i$ is open (is in T) assuming $A_i \in T$.

Recall,
$X = \mathbb{R}$
top $=$ F.C.T.

Show $\mathbb{R} - \bigcap_{i=1}^{n} A_i$ is finite.

$\Vert$ equiv.

$\bigcup_{i=1}^{n} \underbrace{\mathbb{R} - A_i}_{\text{each finite}}$, Finite union of finite sets is finite

this proves $\bigcap_{i=1}^{n} A_i \in T$.

Union Lemma

Idea: Sometimes you can express a 'big' set as union of smaller sets.

Ex. $\mathbb{R}$

$\mathbb{R} = \ldots \cup (-1,1) \cup (0,2) \cup (1,3) \cup (2,4) \cup (3,5) \cup (4,6) \cup \ldots$

Union lemma: Let X be a set & C a collection of subsets of X. [If $\forall x \in X$ $\exists$ C_x s.t. $x \in C_x$] then

$$X = \bigcup_{x \in X} C_x .$$

proof:

To prove two sets are equal show they're subsets of each other

$\boxed{\subset}$ If $x \in X$ then $x \in \bigcup_{x \in X} C_x$. By hypothesis:

$\exists$ some C_x s.t. $x \in C_x$, thus: $x \in \bigcup_{x \in X} C_x$

$\boxed{\supset}$ Show $\bigcup_{x \in X} C_x \subset X$ The union of subsets of X must live in X.

Neighborhood Theorem

nbhd of x is some open set containing x

Theorem: The subset $U \subset X$ is open

$$\Longleftrightarrow$$

in   such that   a subset of

$$\forall\, x \in X \quad \exists\, U_x \ \text{s.t.}\ \ x \in U_x \subset U.$$

For every   there exists   in

EX

$\mathbb{R}^2$

open