Wed. Week 2

1.2 (Bases)

Finish Proof
From Monday ...

$T = \{$all open subsets of $X\}$, $U \in T$

Nbhd Theorem: Let $U \subset X$, $X =$ top. space.

$$U \text{ is open} \iff \underbrace{\forall x \in U \ \exists \text{ open set (a nbhd) } U_x \text{ with } x \in U_x \subset U.}_{\star}$$

proof:

($\Rightarrow$) If U is open, let $x \in U$ be arbitrary.
Let $U_x = U$. B/c U is assumed to be open
& b/c $U \subset U$ the theorem follows.

($\Leftarrow$) Assume $\star$. Prove U is open. (Show $U \in T$).
Let $x \in U$ be arbitrary. Apply $\star$. Then
$\forall x \in U \ \exists \ U_x$ w/ $x \in U_x \subset U$. So by Union Lemma

$$U = \bigcup_{x \in U} U_x \text{ , w/ each } U_x \text{ open.} \quad (\star)$$

So U is open!

Example:

Let $X = \mathbb{R}$

$U_1 = (-\infty, 1)$

$U_2 = (1, \infty)$

$U_3 = (0, 2)$

Because every real # lives in (at least) one of the U_1, U_2, U_3 the union lemma tells us

$$\mathbb{R} = U_1 \cup U_2 \cup U_3$$

Note: $\mathbb{R} \neq U_1 \cup U_2$ b/c the # $1 \notin U_1$ or $1 \notin U_2$.

Basis for a topology on X

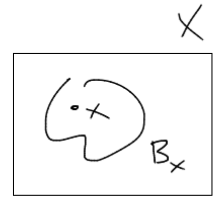
In Lin Alg, the std basis for $\mathbb{R}^2$ is $(1,0)$ & $(0,1)$.

you can express any vector in $\mathbb{R}^2$ via these

$$(7,-3) = 7(1,0) - 3(0,1).$$

In topology the "elements" are subsets

& operations are intersections.



<u>Def'n</u>: Let $\mathcal{B}$ = collection of subsets of X.

$\mathcal{B}$ is a basis if:

1. If $x \in X$ $\exists$ $B_x \in \mathcal{B}$ with $x \in B_x$.

every point in the space lives in SOME basis element

2. If $x \in B_1 \cap B_2$ then $\exists$ $B_3 \subset B_1 \cap B_2$

$x \in B_3$

if the intersections of basis elements contain points, these points are contained in basis elements which themselves are contained in the intersections

<u>Ex.</u> For $X = \mathbb{R}$, $\mathcal{B} = \{$all open intervals (a,b) w/ $a<b\}$

$\mathcal{B}$ is a basis.

- If $x \in 18$ then $18 \in (16,20)$
- For $19 \in (16,20) \cap (17,21)$ the subset $(18,20)$ contains 19 & $(18,20) \subset (16,20) \cap (17,21)$.

$17.5 \in$ ↑ we'd use $(17,18)$ to contain 17.5 yet lie in the intersection

Given a set X, basis $\mathcal{B}$ you get a topology by:
U is open if U is a union of basis elts.

The Upper Limit Topology on $\mathbb{R}$

$X = \mathbb{R}$

$\mathcal{B} = \{(a, b] \subset \mathbb{R} \mid a < b\}$.

By def'n: any union of $B_i \in \mathcal{B}$ is open.

Ex: $\mathbb{R}$ is open b/c. $R_1 = \bigcup_{n=1}^{\infty} (0, n] = (0, \infty)$ is open in U.L.T.

$R_2 = \bigcup_{n=0}^{-\infty} (n, 1] = (-\infty, 1]$ is open in U.L.T.

$\mathbb{R} = R_1 \cup R_2$ is open

Ex $(3, 8]$ is open

Ex $\bigcup_{n=1}^{\infty} (1, 3 - \frac{1}{n}] = (1, 3)$ → open! (not basis elts)

$(1,2) \cup (1, 2.5) \cup (1, 2.\overline{6}) \cup \ldots$