Week 5 Monday

Homework #3 Due this Friday

Today: (the boundary of a set)

<u>Def'n</u>: For a set $A \subset X$. $\partial A = \bar{A} - \mathring{A}$.

Ex. $A = \{1,2,3\}$.
$X = \mathbb{R}$ w/ std. top.

$$\left.\begin{array}{l} \\ \end{array}\right\} \quad \begin{array}{l} \partial A = A \\ \bar{A} = A \quad (A \text{ is closed}) \\ \mathring{A} = \phi \end{array}$$

<u>Ex</u>. $A = \mathbb{R}$, $X = \mathbb{R}$, w/ std top. $\partial A = \phi$

$\overline{\mathbb{R}} = \mathbb{R}$

$\mathring{\mathbb{R}} = \mathbb{R}$

<u>Ex</u>. $(0,1) = A$. $\partial A = \{0,1\}$

$\bar{A} = [0,1]$, $\mathring{A} = A$

General Fact: the boundary of a set is closed.

thm: $x \in \partial A \iff$ every nbhd of x intersects A & every nbhd intersects $X - A$.

(every nbhd intersects both A & $X-A$)

proof:

$\Rightarrow$ Assume $x \in \partial A = \bar{A} - \mathring{A}$. So x is in the closure.

By Theorem 2.5 (our classification for closure)

every nbhd of intersects A. (We're 1/2 done!).

B/c $x \notin \mathring{A}$ no nbhd of x is contained in A.

thus each nbhd of x intersects the complement of A. $\boxtimes$

$\Leftarrow$ Now assume every nbhd of x intersects A. So $x \in \bar{A}$.

Now show $x \notin \mathring{A}$. Since, by assumption every nbhd of x intersects the complement, no nbhd is contained completely in A. thus $x \in \mathring{A}$.

Reading Check

1. Prove: If $A \subset C$ w/ C closed, then $\bar{A} \subset C$.

<u>Hint</u>: $U_1 \cap U_2 \cap U_3 \cap \ldots \subset U_1$

(a) Prove if $A \subset B$ then $\bar{A} \subset \bar{B}$.

Thm. 2.6

(i) $\mathrm{Int}(X - A) = X - \bar{A}$

why is $X - \bar{A} \subset X - A$?

$\mathring{A} \subset A \subset \bar{A} \Rightarrow X - \mathring{A} \supset X - A \supset X - \bar{A}$

Proof of (i). We prove that $X - \mathrm{Cl}(A) \subset \mathrm{Int}(X - A)$ and $\mathrm{Int}(X - A) \subset X - \mathrm{Cl}(A)$. First, note that $\mathrm{Cl}(A)$ is closed and contains A, and therefore $X - \mathrm{Cl}(A)$ is an open set contained in $X - A$. It follows by Theorem 2.2(i) that $X - \mathrm{Cl}(A) \subset \mathrm{Int}(X - A)$. I

To prove that $\mathrm{Int}(X - A) \subset X - \mathrm{Cl}(A)$, let $x \in \mathrm{Int}(X - A)$ be arbitrary. Note that $\mathrm{Int}(X - A)$ is disjoint from A, and therefore x is in an open set that is disjoint from A. By Theorem 2.5, it follows that $x \notin \mathrm{Cl}(A)$; hence, $x \in X - \mathrm{Cl}(A)$. Thus, $\mathrm{Int}(X - A) \subset X - \mathrm{Cl}(A)$. II

Since we have shown that both $X - \mathrm{Cl}(A) \subset \mathrm{Int}(X - A)$ and $\mathrm{Int}(X - A) \subset X - \mathrm{Cl}(A)$ hold, we now have $\mathrm{Int}(X - A) = X - \mathrm{Cl}(A)$, as we wished to show. ■