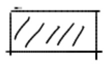
In $(\mathbb{R}^2, \text{std})$ Prove:

$$\overline{(a,b)\times(c,d)} = [a,b]\times[c,d]$$

So $X = (a,b)\times(c,d)$

① show $\underbrace{\overline{(a,b)\times(c,d)}}_{\overline{X}} \subset \underbrace{[a,b]\times[c,d]}_{Y}$

<u>Fact:</u> Y is closed , complement is open

Does Y contain X? — Yes. So Y is a closed set containing X

$$X \subset \overset{\text{closed}}{Y} \implies \overline{X} \subset Y$$

② show $Y \subset \overline{X}$.

Use <u>main theorem</u>: show any nbhd of any point in Y intersects X.

Reading Check

Dense: A set $A \subset X$ is dense in X if $\overline{A} = X$.

Every open set in X intersects A.

Ex $(\mathbb{R}^2, \text{std})$ looking for a non-trivial dense set.

(First, look to $\mathbb{R}$ for inspiration)

—o— A

- In $\mathbb{R}$, the following is dense. $\mathbb{R} - \{0\}$.

  $A = \mathbb{R} - \{0\}$. Every open set in $\mathbb{R}$ intersects A.

In contrast to

- $\mathbb{Z}$ = integers in $\mathbb{R}$, is not dense

  $(\frac{1}{2}, \frac{3}{4})$ is an open set in $\mathbb{R}$, disjoint from $\mathbb{Z}$.

- $\mathbb{Q}$ is dense in $(\mathbb{R}, \text{std})$

EX. $(\mathbb{R}, \text{F.C.T.})$

open sets are infinite and miss only a finite # of points

① Is $x = 3$ dense in this topology? No.

$U = \mathbb{R} - \{3, 4, 5\}$ is open & disjoint from 3.

② $\underbrace{U = \mathbb{R} - \mathbb{Q}}_{\text{infinite}}$

Any open set in F.C.T. $\underbrace{\mathbb{R} - \{x_1, x_2, \dots, x_n\}}_{\text{misses only finitely many}}$

$\Rightarrow$ There is overlap

③ $\mathbb{Z}$ is dense in F.C.T

B/c Any open set in F.C.T.

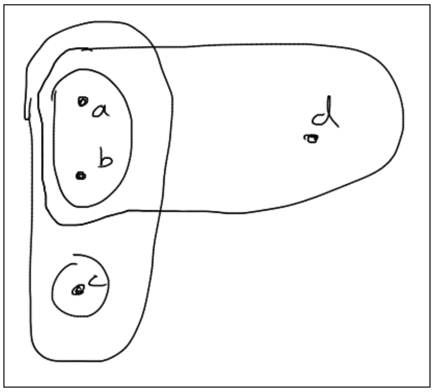
$\boxed{U = \mathbb{R} - \{x_1, \dots, x_n\}}$

Claim $U \cap \mathbb{Z} \neq \phi$. If so $\mathbb{Z}$ is dense b/c U is arbitrary.

($\mathbb{Z}$ will then intersect every open set)

Let $S = \{1, 2, 3, \dots, n, n+1\}$. There must be at least one member of $S \cap U$. Clearly this also lives in $\mathbb{Z}$.

$\Rightarrow U \cap \mathbb{Z} \neq \phi$.



Looking For Dense Sets

sets that lie in every open set.

1. $\{d\}$ is not dense b/c $\{a, b\}$ is open & disjoint from d

2. $\{d, a, c\}$ is dense

3 $\{b, c\}$ is dense.

Keeping the different topologies straight

$(\mathbb{R}, \text{lower limit}) = \mathbb{R}_\ell$ $\subset$ $(\mathbb{R}, \text{std})$

basis includes its lower limit

(Finer than std)

<u>open sets!</u>

(a,b), $(-\infty, a)$, (a, ∞)
(and unions)

$[a,b)$, $(-\infty, a)$, $[a, \infty)$

Notice opens are std open sets plus ones that include their lower limit

$$\bigcup_{n=1}^{\infty} [a+\tfrac{1}{n}, b) = (a,b)$$

<u>Ex</u> $A = (0,1]$ not closed b/c it's complement must include its upper limit. (the aren't open)

$\mathring{A} = (0,1)$, $\bar{A} = [0,1]$

Reading Check

($\mathbb{R}$, discrete)

$A = [0,1) \longrightarrow A = \mathring{A}$

1. A is open (def'n)
2. A is closed (It's complement is some set) Any set is open.

$\longleftrightarrow A = \overline{A}$

$A = (1,5]$

$(\mathbb{R}, F.C.T)$: $\overline{A} = A \cup A'$

$\mathbb{R} = A \cup A'$

Closed Sets in F.C.T.

Any set w/ only a finite # of points or $\mathbb{R}$ or ϕ.

$\boxed{A' = \mathbb{R}}$

$(-\infty, 0] \cup (6, \infty)$

$(0, 6)$