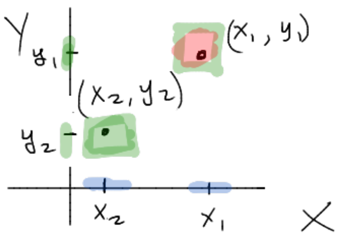
Wed - Week 8

3.18 If X, Y are Hausdorff spaces then so is $X \times Y$.

Important Figure



Let $(x_1, y_1) \in X \times Y$
$(x_2, y_2) \in X \times Y$.

Hausdorff assumption $\Rightarrow$

open sets $U_1, U_2 \in X$ w/ $x_1 \in U_1$, $x_2 \in U_2$
$U_1 \cap U_2 = \phi$

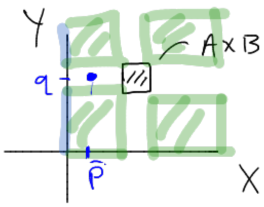
open sets $V_1, V_2 \in Y$ w/ $y_1 \in V_1$, $y_2 \in V_2$
$V_1 \cap V_2 = \phi$

Claim: $\underset{(x_1, y_1)}{U_1 \times V_1} \cap \underset{(x_2, y_2)}{U_2 \times V_2} = \phi$.

(If not $(p, q) \in U_1 \times V_1 \bigcap U_2 \times V_2$ then $p \in U_1 \cap U_2$ & $q \in V_1 \cap U_2$ $\otimes$.

3.19 A closed in X, B closed in Y $\Longrightarrow$ $A \times B$ is closed in $X \times Y$

<u>Lemma</u>: For $A \subset X$, $B \subset Y$ $\quad (X \times Y) - (A \times B) \supset (X - A) \times (Y - B)$



<u>proof</u>:
let $(p,q) \in (X-A) \times (Y-B)$
so $p \notin A$, $q \notin B$.
thus $(p,q) \notin A \times B$
and $(p,q) \in X \times Y$.

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continuing w/ 3.19. Assume A, B are closed in X, Y (resp)
show $A \times B$ is closed, equiv $X \times Y - (A \times B)$ is open.

Let $(p,q) \in X \times Y - (A \times B)$. You know that <u>either</u> $p \notin A$ or $q \notin B$.
Assume $p \notin A$. Then $p \in X - A$, w/ A closed so $X - A$ is open.
$q \in Y$ and Y is open in Y & thus $(X-A) \times Y$ is the product of an open set in X with open set in Y, so $(X-A) \times Y$ is open in the product topology $X \times Y$

<u>AND</u> $(X-A) \times Y$ is contained in $(X \times Y) - (A \times B)$

Since (p,q) is arbitrary, $X \times Y - (A \times B)$ is open, then $A \times B$ is closed

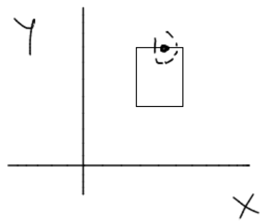
3.20 Show if $A \subset X$, $B \subset Y$ then

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

Note: Thm: $\mathrm{Int}(A \times B) = \mathring{A} \times \mathring{B}$.

$\boxed{\subset}$ Let $(p,q) \in \overline{A \times B}$. Every open nbhd of (p,q) intersects $A \times B$.

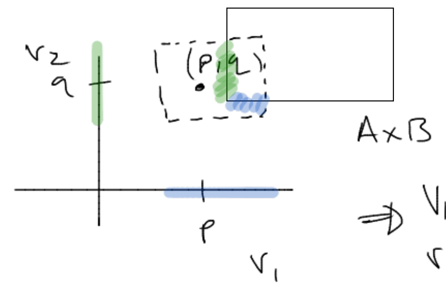
Let U be an open nbhd of (p,q), then



$\exists$ open sets $V_1 \subset X$, $V_2 \subset Y$ of p, q respectively such that

$(p,q) \in V_1 \times V_2 \subset U$

Assumption $V_1 \times V_2 \cap A \times B \neq \emptyset$.



$\Rightarrow V_1 \cap A \neq \emptyset$
$V_2 \cap B \neq \emptyset$
$\Rightarrow p \in \overline{A}, q \in \overline{B}$

$\Rightarrow (p,q) \in \overline{A} \times \overline{B}$.

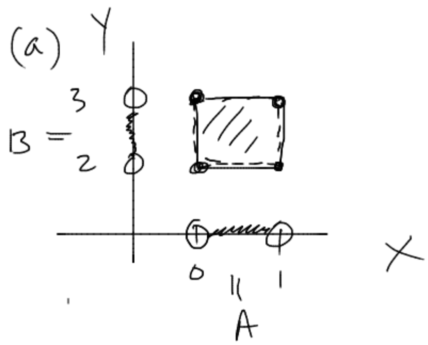
$\boxed{\supset}$ Similar Idea

3.22. Suppose that $A \subset X$ and $B \subset Y$.

(a) Provide an example demonstrating that $\partial(A \times B) = \partial(A) \times \partial(B)$ does not hold in general.

(b) Derive and prove a relationship expressing $\partial(A \times B)$ in terms of $\partial(A)$, $\partial(B)$, A, and B.

(a)



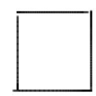
$A \times B$ = open rectangle

$\partial A = \{0,1\}$

$\partial B = \{2,3\}$

$\partial A \times \partial B = \{(0,2),(0,3),(1,2),(1,3)\}$

$\partial(A \times B) =$



Maybe useful (b)

- $\partial A = \bar{A} - \mathring{A}$, $\partial B = \bar{B} - \mathring{B}$
- $\overline{A \times B} = \bar{A} \times \bar{B}$
- $\operatorname{Int}(A \times B) = \mathring{A} \times \mathring{B}$

- Lemma: $A \subset X$, $B \subset Y$

  $\Rightarrow (X - A) \times (Y - B) \subset (X \times Y) - (A \times B)$