Friday – Week 9

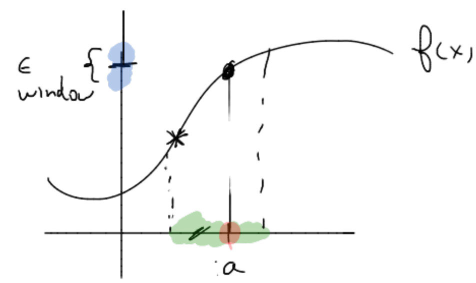
Recall:

$f: X \longrightarrow Y$ is cts if $U \subset Y$ is open then $f^{-1}(U)$ is open in X.

<u>Analysis Def'n:</u>

$f: \mathbb{R} \longrightarrow \mathbb{R}$ is cts at $x = a$ if $\forall\, \epsilon > 0$ $\exists\, \delta > 0$ s.t.

if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

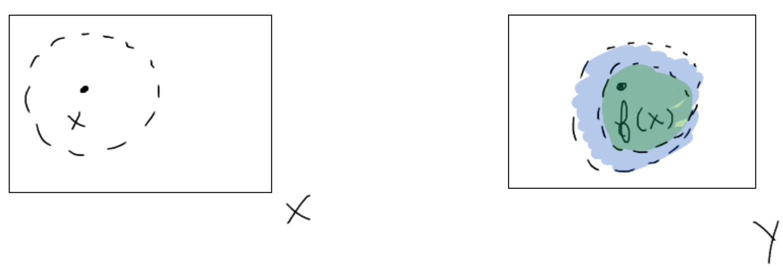


ϵ – window size in range about $f(a)$

Translation of $\epsilon - \delta$ Analysis Def'n to open sets

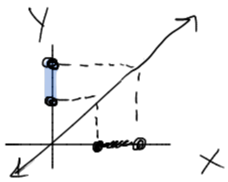
X, Y are $\mathbb{R}$, std topologies:

$f: X \to Y$ is cts if $\forall\, x \in X$ & $\forall$ open set U in Y containing $f(x)$ $\exists$ open set V in X such that $f(V) \subset U$



Thm: $f: X \to Y$ is cts iff the preimage of any closed set in Y is closed in X.

Ex $y = x$



Ex $f(x) = x^2 + 1$, $f: \mathbb{R} \to \mathbb{R}$

$$f^{-1}([0,10]) = [-3,3]$$

Proof: Homework. Use the following

Thm: f cts $\Rightarrow$ $f(\bar{A}) \subset \overline{f(A)}$

Ex. $f(x) = x^2 + 1$ is a poly (thus cts.) (std)

Let $A = [0,1)$.

$f(A) = [1,2)$

$f(\bar{A}) = f([0,1]) = [1,2]$

$\overline{f(A)} = \overline{[1,2)} = [1,2]$

$$f^{-1}(Y - A) = f^{-1}(Y) - f^{-1}(A)$$

$\boxed{\subset}$ if $x \in f^{-1}(Y-A)$ then $f(x) \notin A$ then $x \notin f^{-1}(A)$
$\Rightarrow x \in f^{-1}(Y) - f^{-1}(A)$

$\boxed{\supset}$ same idea

Proof: Let $y \in f(\bar{A})$. Let $x \in f^{-1}(y) \subset \bar{A}$. So every nbhd of x intersects A.

Note $\overline{f(A)}$ is closed in Y, $Y - \overline{f(A)}$ is open in Y.

So $f^{-1}(Y - \overline{f(A)})$ is open in X b/c f is cts

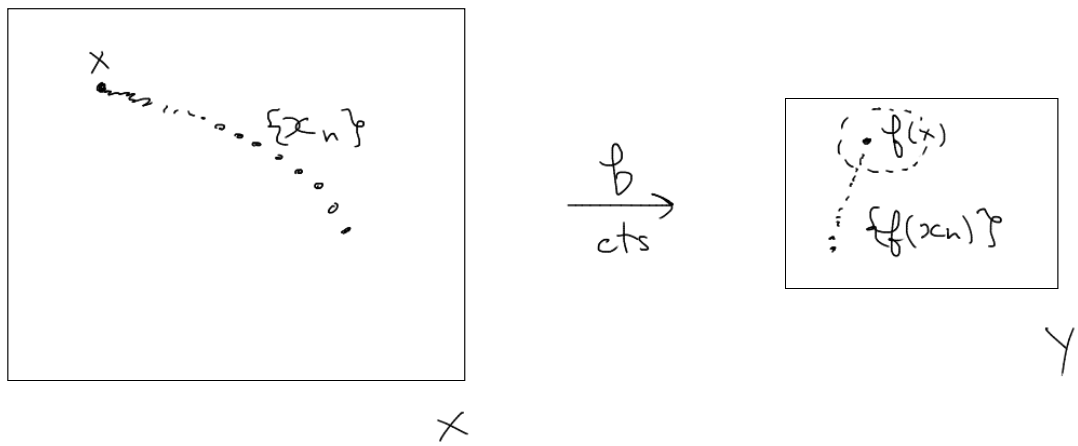
$$= f^{-1}(Y) - f^{-1}(\overline{f(A)})$$

$$= X - f^{-1}(f(\bar{A}))$$ is open.

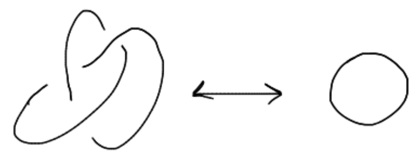
If $\exists$ open nbhd, V, of y disjoint from $f(A)$ then $f^{-1}(V)$ is open set containing x disjoint from A. Contradicts red line above.

Thm: Continuous functions preserve convergence.

Idea:



Ex **Fact** $\left\{\frac{1}{n^2}\right\} \xrightarrow{\text{converges}} 0$



Fact: $\sin(x)$ is continuous.

$$\left\{\sin\left(\frac{1}{n^2}\right)\right\} \xrightarrow[\text{converges}]{} \sin(0)$$