Week 10 — Monday

- More Euler Sums
- Last Time: used $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ to show $\frac{1}{5!} = \frac{1}{120}$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{5!} + \cdots$$

sums of reciprocals of $\square$'s

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

Key:

$$\frac{\sin x}{x} = \left[1 - \frac{x^2}{\pi^2}\right] \cdot \left[1 - \frac{x^2}{4\pi^2}\right] \cdot \left[1 - \frac{x^2}{9\pi^2}\right] \cdot \left[1 - \frac{x^2}{16\pi^2}\right] \cdots \quad (\star)$$

Ex Evaluate $\frac{\sin x}{x}$ as above at $x = \frac{\pi}{2}$.

$$\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} = \left[1 - \frac{\left(\frac{\pi}{2}\right)^2}{\pi^2}\right] \cdot \left[1 - \frac{\left(\frac{\pi}{2}\right)^2}{4\pi^2}\right] \cdot \left[1 - \frac{\left(\frac{\pi}{2}\right)^2}{9\pi^2}\right] \cdot \left[1 - \frac{\left(\frac{\pi}{2}\right)^2}{16\pi^2}\right] \cdots$$

$$= \left[1 - \frac{1}{4}\right] \cdot \left[1 - \frac{1}{16}\right] \cdot \left[1 - \frac{1}{36}\right] \cdot \left[1 - \frac{1}{64}\right] \cdots$$

$$= \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \frac{63}{64} \cdots$$

$$\frac{2}{\pi} = \frac{3 \cdot 15 \cdot 35 \cdot 63}{4 \cdot 16 \cdot 36 \cdot 64} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}$$

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9} \cdots$$

John Wallis,

Next, we explore how Euler discovered a relationship b/w 4^{th} powers of π & inverses.

$$\frac{\sin x}{x} = \left[1 - \frac{x^2}{\pi^2}\right]\cdot\left[1 - \frac{x^2}{4\pi^2}\right]\cdot\left[1 - \frac{x^2}{9\pi^2}\right]\cdot\left[1 - \frac{x^2}{16\pi^2}\right] \cdots$$

When we expand and collect all terms the 4^{th} degree pieces look like:

$$\left(1 - \frac{x^2}{a}\right)\left(1 - \frac{x^2}{b}\right) = 1 + \left(-\frac{1}{a} - \frac{1}{b}\right)x^2 - \frac{1}{ab}x^4$$

$$(1 - ax^2)(1 - bx^2) = 1 - 2ab\,x^2 + ab\,x^4$$

$$ab = \frac{1}{2}\Big[\underbrace{(a+b)^2 - a^2 - b^2}_{\substack{a^2+2ab+b^2-a^2-b^2 \\ =2ab}}\Big]$$

$$(1 - ax^2)(1 - bx^2)(1 - cx^2) = 1 + (\text{stuff})(x^2) - (\underbrace{ab + ac + bc}_{})x^4 - abc\,x^6$$

$$\boxed{\frac{1}{2}\left((a+b+c)^2 - a^2 - b^2 - c^2\right)}$$

applying this gen'l non-sense to $\frac{\sin x}{x}$ we see: $a = \frac{1}{\pi^2},\ b = \frac{1}{4\pi^2},\ c = \frac{1}{9\pi^2}, \ldots$

?

$$\frac{\sin x}{x} = 1 - \Big(\underbrace{\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots}_{\frac{1}{6}}\Big)x^2 - \underbrace{\left(\frac{1}{2}\left[\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)^2 - \frac{1}{\pi^4} - \frac{1}{16\pi^4} - \frac{1}{81\pi^4} - \cdots\right]\right)}_{120} - \cdots$$

$$\frac{1}{120} = +\left(\frac{1}{2}\left[\left(\frac{1}{36} + \frac{1}{\pi^4}\left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots\right)\right)\right]\right.$$

$$= +\frac{1}{72} + \frac{1}{2\pi^4}\left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots\right)$$

$$\underbrace{\frac{1}{120} - \frac{1}{72}}_{} = +\frac{1}{2\pi^4}\left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} \cdots\right)$$

$$\left(\frac{1}{180}\right)\cdot 2\pi^4 = \left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots\right)$$

$$\frac{\pi^4}{90} = \left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots\right)$$

Reading! the Great theorem can be used to find other series.

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \dots + \frac{1}{(2k)^2} = \boxed{\frac{\pi^2}{24}}$$

Factor $1/4$:

$$\frac{1}{4}\Big(\underbrace{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots}_{\frac{\pi^2}{6}}\Big) = \frac{\pi^2}{24}$$

reproduce this

using:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$\Rightarrow$ reproduce "Great Thm"

also, the "odd" case:

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \dots =$$

$$\underbrace{1 + \cancel{\frac{1}{4}} + \frac{1}{9} + \cancel{\frac{1}{16}} + \frac{1}{25} + \dots} - \Big(\cancel{\frac{1}{4}} + \underbrace{\cancel{\frac{1}{16}} + \frac{1}{64}} + \dots\Big)$$

$$= 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}$$

Gauss's Childhood:

early:

What's the sum of the 1^{st} 100 integers?

$$S = 1 + 2 + 3 + 4 + 5 + \dots + 98 + 99 + 100$$

$$+ \quad S = 100 + 99 + 98 + 97 + 96 + \dots + 3 + 2 + 1$$

$$2 \cdot S = 101 + 101 + 101 + 101 + 101 + \dots + 101 + 101 + 101$$

$$2S = 100(101)$$

$$S = 50(101) = 5050$$

more generally:

$$\frac{21 \cdot 22}{2} = 231$$

$$\begin{array}{r} 21 \\ 11 \\ \hline \end{array}$$

$$S = 1 + 2 + 3 + \dots + n$$

$$+ \quad S = n + (n-1) + (n-2) + \dots + 1$$

$$2 \cdot S = (n+1) + (n+1) + (n+1) + \dots + (n+1)$$

$$= n(n+1)$$

$$S = \frac{n(n+1)}{2}$$