$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

using similar methods: Euler derived

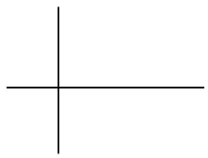
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

These formular are used in your calculator to compute (approximations) of $\sin(3)$

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$$



Roots of Polys:

$\updownarrow$

Factors

$x^2 - 7x + 12$

Factors: $(x-3)(x-4)$

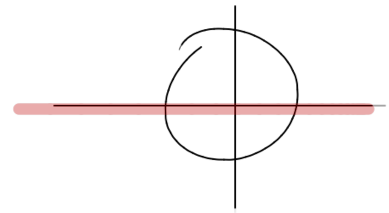
Roots: $x = 3$, $x = 4$

$5x^2 - 35x + 60$

Roots: 3, 4

$5(x-3)(x-4)$

sin x



$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \dots$$

Roots?

$\cancel{0}, \pi, n\pi, \pm n\pi$ } Roots of $\sin x$

Euler knew this, assumed that roots <—> factors

$$= C\underbrace{(x-\pi)(x+\pi)}\underbrace{(x-2\pi)(x+2\pi)}(x-3\pi)\dots$$

$$= C(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)(x^2-16\pi^2)\dots$$

$$= \underbrace{C(-\pi^2)(-4\pi^2)(-9\pi^2)\dots}_{=1}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\dots$$

$$(x^2-\pi^2) = -\pi^2\left(1-\frac{x^2}{\pi^2}\right)$$

$$-\pi^2\left(1-\left(\frac{x}{\pi}\right)^2\right)$$

Euler knew

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\frac{\sin x}{x} = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\dots$$

$$= 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \dots\right)x^2 + \dots$$

$$(1-ax^2)(1-bx^2)(1-cx^2)$$

$$1-(a+b+c)x^2+\dots$$

so...

Combining w/ series above, equating coeffs of x^2

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \dots\right) = -\frac{1}{3!}$$

$$\frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots\right) = \frac{1}{6}$$

$$\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots\right) = \frac{\pi^2}{6}$$

By doing the same thing for x^4 term Euler found

$$1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \dots + \frac{1}{n^4} = \frac{\pi^4}{90}$$

Is there a formula to be found from the 4th power?

$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(\ldots\right.$$

Finite Example: $(1-ax^2)(1-bx^2) = (1-bx^2-ax^2+abx^4) = 1-(a+b)x^2+abx^4$

maybe product

$$(1-ax^2)(1-bx^2)(1-cx^2) = (1-(a+b)x^2+abx^4)(1-cx^2)$$

$$= 1-(a+b)x^2+abx^4-cx^2+(a+b)cx^4-abcx^6$$

$$= 1-(a+b+c)x^2+(ab+ac+bc)x^4-abcx^6$$

Pattern:

$$(a+b+c)^2 = (a+b+c)(a+b+c) = a^2+ab+ac+ba+b^2+bc+ac+bc+c^2$$

$$= a^2+2ab+2ac+2bc+b^2+c^2$$

$$= a^2+b^2+c^2+2ab+2bc+2ac$$

So

$$\frac{1}{2}\left((a+b+c)^2-(a^2+b^2+c^2)\right) = \text{coef. of degree 4 term!}$$

generalizing to ∞ products:

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(\ldots\right. =$$

$$= 1-\frac{1}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\right)x^2+\frac{1}{2}\left[\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\ldots\right)^2-\left(\left(\frac{1}{\pi^2}\right)^2+\left(\frac{1}{4\pi^2}\right)^2+\left(\frac{1}{9\pi^2}\right)^2+\ldots\right]x^4$$

$$\frac{1}{2}\left[\left(\frac{1}{\pi^2}\right)^2\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\right)^2-\left(\frac{1}{\pi^2}\right)^2\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\right)\right]x^4$$

$$= \frac{1}{2\pi^4}\left[\left(\frac{\pi^2}{6}\right)^2-\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\right)\right]x^4$$

$$= \left[\frac{1}{72}-\frac{1}{2\pi^4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\right)\right]x^4$$

Relationship B/W Reciprocals of 4th powers & 4th power of π.

must $= \frac{1}{5!} = \frac{1}{120}$

so $\frac{1}{2\pi^4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\frac{1}{n^4}\right) = \frac{1}{72}-\frac{1}{120} = \frac{1}{180} \Rightarrow \boxed{\sum_{k=0}^{\infty}\frac{1}{k^4} = \frac{1}{180}\cdot 2\pi^4 = \frac{\pi^4}{90}}$