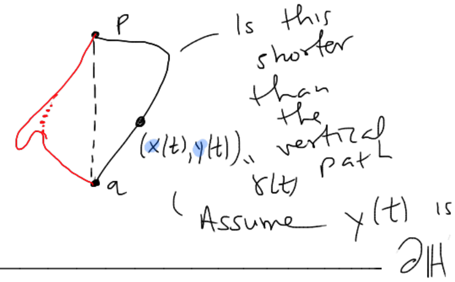
Mon. wk 11

Prop. Euclidean lines $\perp$ to ∂H are geodesics (hyperbolic line) (Upper Half-Plane model)



Recall, when p, q lie on a vertical line

$$d_H(p,q) = \ln(p) - \ln(q) = \ln\left(\frac{p}{q}\right)$$

(Assume $y(t)$ is an increasing function.

$$s = \text{length of } \gamma(t) = \int_{t_0}^{t_1} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y}\, dt \geq \int_{t_0}^{t_1} \frac{\sqrt{0 + \left(\frac{dy}{dt}\right)^2}}{y}\, dt = \int_{t_0}^{t_1} \frac{\left|\frac{dy}{dt}\right|}{y}\, dt$$

$$\geq \int_{t_0}^{t_1} \frac{\frac{dy}{dt}}{y}\, dt = \int_{t_0}^{t_1} \frac{dy}{y} = \ln|y| \Big|_{t_0}^{t_1} = \ln(y(t_1)) - \ln(y(t_0))$$

$$= \ln\left(\frac{y(t_1)}{y(t_0)}\right) = \text{length of vertical path}$$

$$(x,y) \xrightarrow{\Phi} (u(x,y),\, v(x,y))$$

∂H

Φ is isometry if

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + dy^2}{y^2}$$

In Euclidean Geom

$$\Phi(x,y) = (\underbrace{x+1}_{u},\, \underbrace{y+2}_{v})$$

$$du^2 + dv^2 = dx^2 + dy^2$$

Example:

$$\Phi_a(x,y) = (x+a,\, y)$$ is an isom. (see above)

$$\varphi_b(x,y) = (\underbrace{x}_{u},\, \underbrace{y+b}_{v})$$

$u = x$

$du = dx$

$v^2 = y^2 + 2yb + b^2$

$v = y + b$

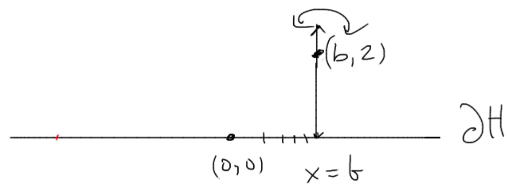
$dv = dy$

Not an isom.

$$\frac{dx^2 + dy^2}{y^2} = \frac{du^2 + dv^2}{y^2 + 2yb + b^2}$$

Ex. $R_b(x,y) = (u,v) = (2b - x, y)$

$du = -dx$, $dv = dy$ | $du^2 = dx^2$, $dv^2 = dy^2$ | y is unchanged $\Rightarrow$ Isom.



Next, we want to see that "reflections" across non-straight geodesics are isometries.

$$u = \frac{x}{x^2+y^2}$$

$$du = \frac{r^2 dx - x(2x\,dx + 2y\,dy)}{r^4}$$

$$du = \frac{(r^2 - 2x^2)dx - (2xy)dy}{r^4}$$

Important Map: $\Phi(x,y) = (u,v) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$

Set $r^2 = x^2 + y^2$

show H-metric is preserved

$$\frac{du^2 + dv^2}{v^2} = \frac{\left((r^2-2x^2)dx - (2xy)dy\right)^2/r^8 + \left((-2xy)dx + (r^2-2y^2)dy\right)^2/r^8}{\left(\frac{y^2}{r^4}\right)}$$

$$dv = \frac{r^2 dy - y(2x\,dx + 2y\,dy)}{r^4} \qquad v = \frac{y}{x^2+y^2}$$

$$= \frac{(-2xy)dx + (r^2-2y^2)dy}{r^4}$$

$-4xy(y^2 - x^2)$

$-4xy(x^2+y^2-2x^2)$

$$= \frac{1}{r^4 y^2}\Big[(r^4 - 4r^2x^2 + 4x^4)dx^2 - 4xy(r^2-2x^2)\,dx\,dy + 4x^2y^2\,dy^2$$

$$4x^2y^2\,dx^2 \quad \underbrace{(-4xy)(r^2-2y^2)}\,dx\,dy \quad + (r^4 - 4r^2y^2 + 4y^4)\,dy^2\Big]$$

$-4xy(x^2 - y^2)$

$$= \frac{1}{r^4y^2}\Big[r^2(r^2-4x^2) + 4x^4 + 4x^2y^2\,dx^2$$

$x^2+y^2-4x^2$

$r^2(y^2-3x^2) + 4x^2(x^2+y^2)$

$r^2(y^2 - 3x^2 + 4x^2)$

$r^2(y^2+x^2) = r^4$

$r^4 \underbrace{+ 4x^2y^2 - 4(x^2+y^2)y^2}_{=0} + 4y^4$

$-4x^2y^2 \underbrace{- 4y^4 + 4y^4}_{=0}$

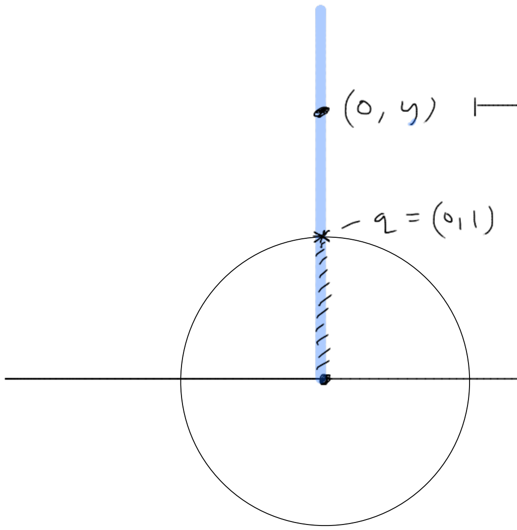
$r^4\,dy^2$

$$= \frac{1}{r^4y^2}\left[r^4\,dx^2 + r^4\,dy^2\right] = \frac{dx^2+dy^2}{y^2}$$

what kind of map is this? ____________

Inversion about unit-circle

$$(x,y) \longmapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$



$$(0,y) \longmapsto \left(\frac{0}{0^2+y^2}, \frac{y}{y^2}\right) = \left(0, \frac{1}{y}\right)$$

$q = (0,1)$

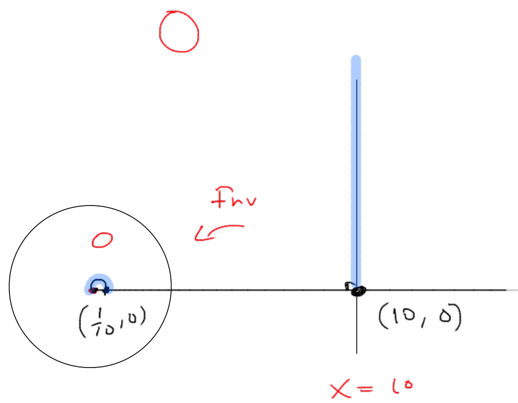
$$d_H((0,1),(0,y)) = \ln(\text{higher } y) - \ln(\text{lower } y) = \ln(y)$$

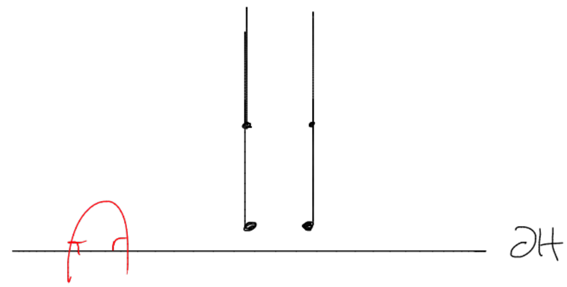
$$d_H\left((0,1),\left(0,\tfrac{1}{y}\right)\right) = \ln(1) - \ln(1/y) = 0 - (\ln(1) - \ln(y)) = \ln(y)$$

Homework: What does inversion about unit circle do to the line y = 2x ?

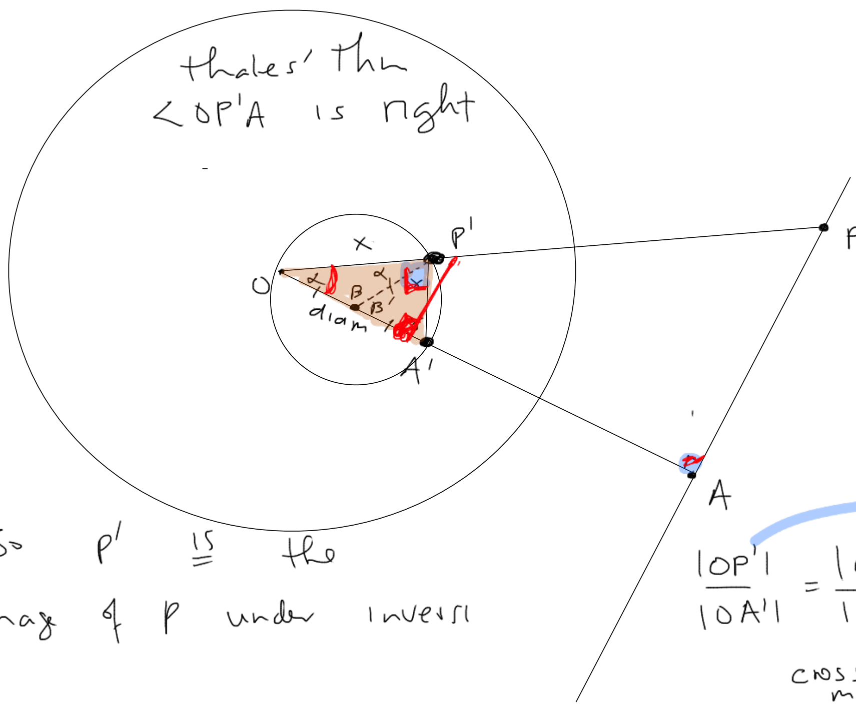
Lemma Let ℓ be a Euclidean line not thru the origin.

The image of ℓ under "inversion about unit circle" is a circle which goes thru origin.





Assume line doesn't intersect u.c.



1. Drop $\perp$ from O to ℓ, forming A
2. Let A' = image of A under A
3. Repeat for point P
4. Form circle Γ w/ center = O, diameter = OA'
5. $\Gamma \cap \overleftrightarrow{OP} = P'$ and claim P' = image of P.

Key: $\triangle OP'A' \sim \triangle OAP$

$$\frac{|OP'|}{|OA'|} = \frac{|OA|}{|OP|} \Rightarrow \frac{x}{1/|OA|} = \frac{|OA|}{|OP|}$$

cross mult $\quad x = \frac{1}{|OP|} = |OP'|$

So P' is the image of P under inversion