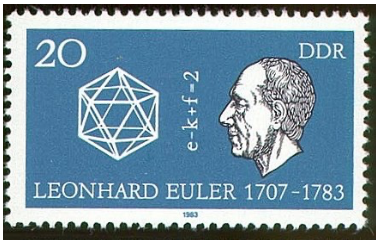
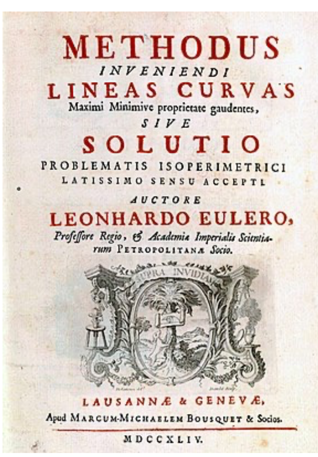
Euler

1. Almost a 1/3 of the math & science created in the 18th century bears his name.
2. Poor eyesight, eventually blind
    a. but he said: “on the bright side, I have fewer distractions”
    b. 1738: Degenerative sight in right eye
    c. 1766: cataract in left eye, failed surgery left him almost totally blind.
3. Two students once disagreed over the result of the sum of 17 terms in a series, (their results differed in the **fiftieth** decimal place).
    a. Euler computed the correct result, in his head, in a few seconds.
4. Like many, disobeyed father (study religion), he took advice of Bernouilli to take up math.
5. It was in St. Petersburg, Russian city, on September 18, 1783, Euler was calculating the ascent of hot air balloons—which at that time were causing a furore in Europe—and argued over dinner with his colleague Anders Johan Lexell about the orbit of the newly discovered planet `. As Condorcet wrote, it was later, while drinking tea and playing with his grandson, when “all of a sudden the pipe that he was smoking slipped from his hand and he ceased to calculate and live.” ~





METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ Socio.

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

Euler's life Timeline

1. Oldest of 4 children
2. At 13, began University of Basel
3. At 16, Masters of Philosophy: compared philosophies of Descartes & Newton
4. At 20, entered Paris Academy prize competition
    a. What's the best way to place the masts on a ship?
        i. Took 2nd place behind Pierre Bouguer - father of naval architecture
    b. Euler entered this competition 15 times (winning 12)
5. At 20, worked Russian Academy of Sciences with Daniel Bernouilli (replacing Nicolaus)
    a. Mastered Russian
    b. Medic in Navy
    c. Had long post at the Academy (physics, math)
6. 1734 (At 28) married Katharina Gsell
    a. 13 children, only 5 survived childhood
7. 1741 (At 34) left Russia
    a. Berlin Academy
8. 1748 (At 41) Text: Introductio in analysin infinotrum
    a. Foundations of mathematical analysis
9. 1755 (At 48) Text: Differential Calculus
10. 1755 (At 48)
    a. Member of Royal Swedish Academy of Sciences
    b. French Academy of Sciences
11. Early 1760's
    a. 200 letters that became
        i. Letters of Euler on different Subjects in Natural Philosophy Addressed to a German Princess
        ii. 
        iii. The popularity of these Letters testifies to Eulers teaching ability (a rarity)
12. 1773 (At 64) His wife died
    a. 3 years later he married her half sister Salome Gsell.

What was Euler not?

▼1. ?

   a. He was not Voltaire.
   b. Euler was a simple, devoutly religious man who never questioned the existing social order or conventional beliefs
   c. He was not a skilled debater.

Basel Problem

Started Bernoulli — finished by Euler

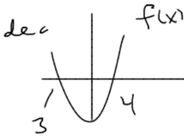
$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots + \; = \; ?$

Tools:

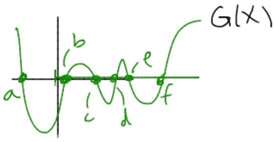
① Euler assumed patterns in finite math continued to infinity

② F.T.A.: Every polynomial (finite) factors

idea



$f(x) = x^2 - 7x + 12$
$= (x-3)(x-4)$



$\Longrightarrow \; G(x) = C(x-a)(x-b)\cdots(x-f)$

③ Factor: $(x-a) = (a-x) = a\left(1 - \frac{x}{a}\right)$

④ $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Ⓐ $$\frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \ldots$$

Ⓑ Roots of $\sin(x)$: $\sin x = 0 \Rightarrow x = 0, n\pi \;\; \forall\, n \in \mathbb{Z}$ So ... Roots of $\frac{\sin(x)}{x} = n\pi \;\; \forall\, n \in \mathbb{Z}$.

Ⓒ $$\frac{\sin(x)}{x} = C\underbrace{(x-\pi)(x+\pi)}_{x^2-\pi^2}\underbrace{(x-2\pi)(x+2\pi)}_{x^2-(2\pi)^2}(x^2-(3\pi)^2)(x^2-(4\pi)^2)(x^2-(5\pi)^2)\cdots$$

$$= \underbrace{C\cdot 36\pi^2 \cdot 25\pi^2 \cdot 9\pi^2 \cdot 4\pi^2 \pi^2}_{=1}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{25\pi^2}\right)\left(1-\frac{x^2}{36\pi^2}\right)$$

B/c $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\frac{\sin(x)}{x} = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{25\pi^2}\right)\left(1-\frac{x^2}{36\pi^2}\right)\cdots$$

$$= 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \frac{1}{25\pi^2}\right)x^2 + \left(\;\ldots\ldots\ldots\ldots\;\right)x^4$$

equating Ⓐ = Ⓒ gives

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \frac{1}{25\pi^2} + \ldots\right)$$

$$\frac{1}{6} = \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots\right)$$

So ...

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots + \frac{1}{n^2} = \frac{\pi^2}{6}$$

Is there a formula to be found from the 4th power?

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(\ldots\right.$$

Finite Example: $(1-ax^2)(1-bx^2) = (1 - bx^2 - ax^2 + abx^4) = 1-(a+b)x^2 + abx^4$

maybe product

$$(1-ax^2)(1-bx^2)(1-cx^2) = (1-(a+b)x^2 + abx^4)(1-cx^2)$$

$$= 1-(a+b)x^2 + abx^4 - cx^2 + (a+b)cx^4 - abcx^6$$

$$= 1-(a+b+c)x^2 + (ab+ac+bc)x^4 - abcx^6$$

Pattern:

$$(a+b+c)^2 = (a+b+c)(a+b+c) = a^2 + ab + ac + ba + b^2 + bc + ac + bc + c^2$$

$$= a^2 + 2ab + 2ac + 2bc + b^2 + c^2$$

$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$

So

$$\frac{1}{2}\left((a+b+c)^2 - (a^2+b^2+c^2)\right) = \text{coef. of degree 4 term!}$$

generalizing to ∞ products:

$$\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(\ldots\right. =$$

$$= 1 - \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\right)x^2 + \frac{1}{2}\left[\underbrace{\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots\right)^2 - \left(\left(\frac{1}{\pi^2}\right)^2 + \left(\frac{1}{4\pi^2}\right)^2 + \left(\frac{1}{9\pi^2}\right)^2 + \ldots\right.}\right] x^4$$

$$\frac{1}{2}\left[\left(\frac{1}{\pi^2}\right)^2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\right)^2 - \left(\frac{1}{\pi^2}\right)^2\left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots\right)\right] x^4$$

$$= \frac{1}{2\pi^4}\left[\left(\frac{\pi^2}{6}\right)^2 - \left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots\right)\right] x^4$$

$$= \left[\frac{1}{72} - \frac{1}{2\pi^4}\left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots\right)\right] x^4$$

Relationship B/W Reciprocals of 4th powers & 4th power of π.

must $= \frac{1}{5!} = \frac{1}{120}$

So $\frac{1}{2\pi^4}\left(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots \frac{1}{n^4}\right) = \frac{1}{72} - \frac{1}{120} = \frac{1}{180} \Rightarrow \boxed{\sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{1}{180} \cdot 2\pi^4 = \frac{\pi^4}{90}}$