# Other Geometrically Constructible Numbers

▼ 1. Definition

    a. Length of constructible segment

    b. Equivalently, x-coordinate of a constructible point

    c. Points are intersections of constructible segments

▼ 2. Given segments with length **a,b** construct segments of length:

    a. **ab**

    b. **a / b**

    c. $\sqrt{a}$

▼ 3. The problems above require these antique tools

    a. Intercept Theorem (Thales)

    b. Geometric Mean Theorem (Euclid)

Intercept Theorem — equivalent to the theorem of Similar triangles ———.

$$\frac{SC}{CD} = \frac{SA}{AB}$$

$$\frac{SC}{SD} = \frac{SA}{SB}$$

$$\frac{SD}{SC} = \frac{SB}{SA}$$

and

$$\frac{SA}{SB} = \frac{AC}{BD}$$

Note:
$$\overleftrightarrow{AC} \parallel \overleftrightarrow{DB}$$

thale's
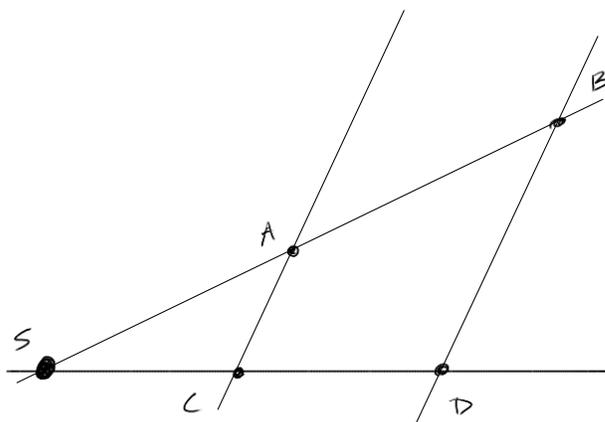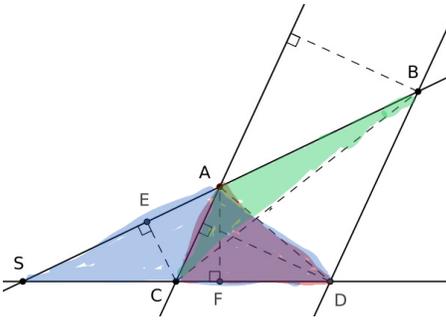


① Blue: base ⎱ two △'s
  Green: height ⎰

$$|\triangle ACD| = |\triangle ABC|$$

② 

and
$$|\triangle SDA| = |\triangle SCB|$$

This follows from above as each triangle contains the same base triangle plus an equivalent additional one.

$$|\triangle SDA| = |\triangle SCB|$$

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**Notation:** For a triangle the vertical bars ($|\ldots|$) denote its area and for a line segment its length.

**Proof:** Since $CA \parallel BD$, the altitudes of $\triangle CDA$ and $\triangle CBA$ are of equal length. As those triangles share the same baseline, their areas are identical. So we have $|\triangle CDA| = |\triangle CBA|$ and therefore $|\triangle SCB| = |\triangle SDA|$ as well. This yields

$$\frac{|\triangle SCA|}{|\triangle CDA|} = \frac{|\triangle SCA|}{|\triangle CBA|} \quad \text{and} \quad \frac{|\triangle SCA|}{|\triangle SDA|} = \frac{|\triangle SCA|}{|\triangle SCB|}$$

Plugging in the formula for triangle areas ($\frac{\text{baseline} \cdot \text{altitude}}{2}$) transforms that into

$$\frac{|SC||AF|}{|CD||AF|} = \frac{|SA||EC|}{|AB||EC|} \quad \text{and} \quad \frac{|SC||AF|}{|SD||AF|} = \frac{|SA||EC|}{|SB||EC|}$$

Canceling the common factors results in:

(a) $\dfrac{|SC|}{|CD|} = \dfrac{|SA|}{|AB|}$ and (b) $\dfrac{|SC|}{|SD|} = \dfrac{|SA|}{|SB|}$

Now use (b) to replace $|SA|$ and $|SC|$ in (a):

$$\frac{\frac{|SA||SD|}{|SB|}}{|CD|} = \frac{\frac{|SB||SC|}{|SD|}}{|AB|}$$

Using (b) again this simplifies to: (c) $\dfrac{|SD|}{|CD|} = \dfrac{|SB|}{|AB|}$ □



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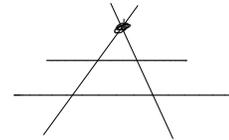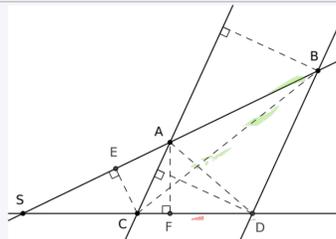
Draw an additional parallel to $SD$ through A. This parallel intersects $BD$ in G.

Then one has $|AC| = |DG|$ and due to claim 1 $\dfrac{|SA|}{|SB|} = \dfrac{|DG|}{|BD|}$ and

therefore $\dfrac{|SA|}{|SB|} = \dfrac{|AC|}{|BD|}$

□



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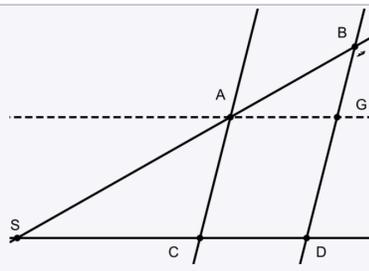
"S"

$$\boxed{\frac{SA}{SB}} = \frac{SC}{CD}$$

"B"

$$\frac{AB}{SB} = \frac{BG}{BD}$$

$$\frac{AB}{SA} = \frac{BG}{GD}$$

⊛

$$\boxed{\frac{SB}{AS}} = \frac{BD}{GD} \Bigg\} \Rightarrow \frac{SA}{SB} = \frac{GD}{BD}$$

$31 \ \frac{1}{9} \ 21 \Rightarrow$ Construct a segment of length $31 \cdot 21 = 651$

$$\frac{|x|}{31} = \frac{21}{1}$$

$$|x| = 21 \cdot 31$$



$21$

$1$

———————— $31$ ——————

————————————————————
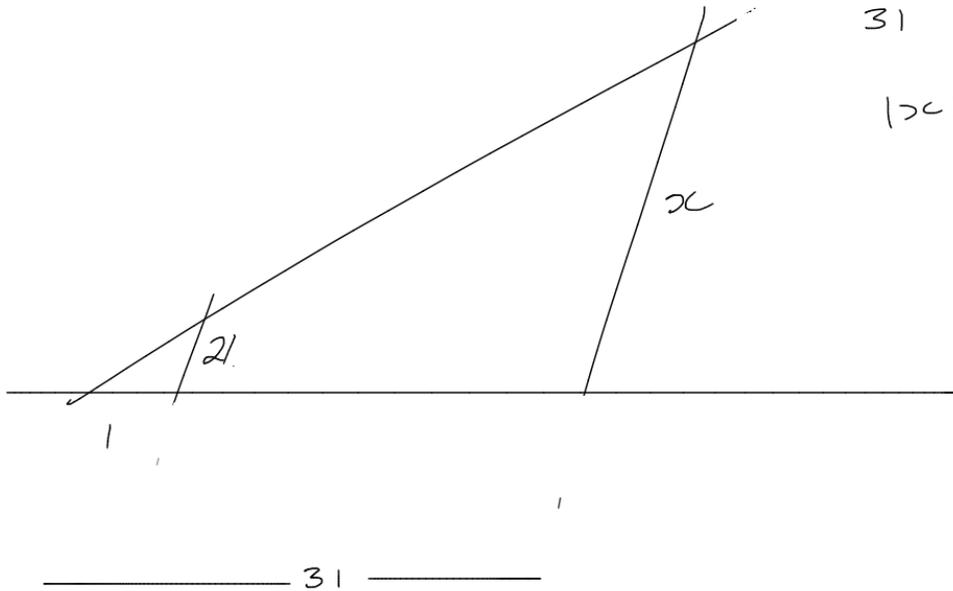
Can you construct a segment of length $\frac{a}{b}$ ?



$a$

$x$

$1$

$$\frac{|a|}{|b|} = \frac{|x|}{1}$$

$$|x| = \frac{|a|}{|b|}$$

——————————————————
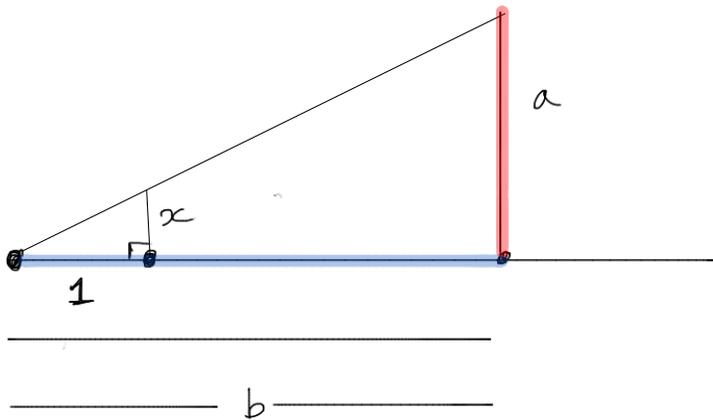
———————— $b$ ——————

squaring the
rectangle
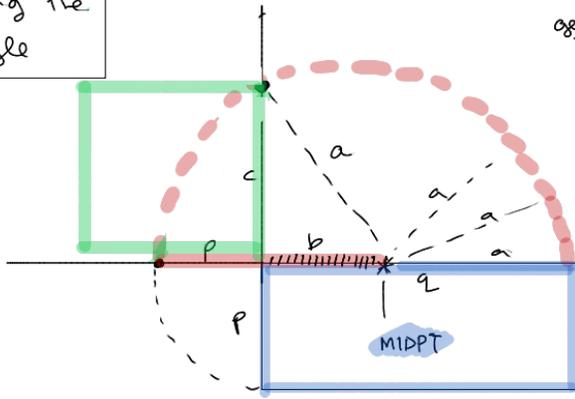
geometry        $p \cdot q$

$$= (a-b)(a+b)$$

$$= a^2 - b^2$$

pythagoras

$$a^2 = c^2 + b^2$$

$$c^2 = a^2 - b^2 \quad (\text{area of square on } c)$$

① start w/ pq rectangle
② circle @ vertex, extend q by p.
③ take midpt, creating radius a
④ raise vertical from p segment, intersect arc @ midpt w/ radius a.

⑤ p = radius − b, q = radius + b, difference of squares
⑥ Pythagoras

The geometric mean theorem

{ use similar △'s to
   show $h = \sqrt{pq}$ }



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$$\frac{p}{h} = \frac{h}{q}$$

$$h^2 = pq$$

$$h = \sqrt{pq}$$



β  α

h

α        q

p'

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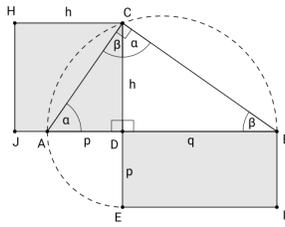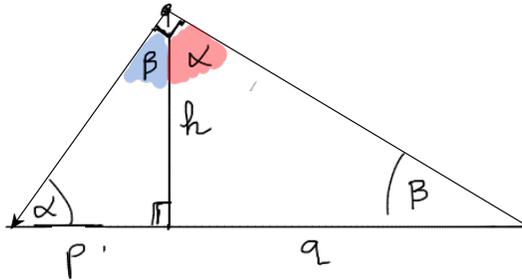
Seg. of length $\sqrt{5}$

$\sqrt{5} = \sqrt{5 \cdot 1}$, set $p = 5$, $q = 1$



Thale's $\Rightarrow 90°$

Geom Mea $\Rightarrow$
$h = \sqrt{5 \cdot 1} = \sqrt{5}$

h

—— 5 ——   1
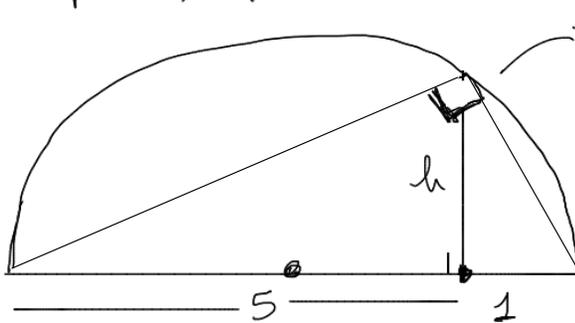
To construct a segment of length root 5, draw a circle
of radius 3, raise a perpendicular off the diameter at 1
unit away from edge.  You raised root 5.  yeah.

# Use Geometric Mean Theorem to construct $\sqrt{p}$
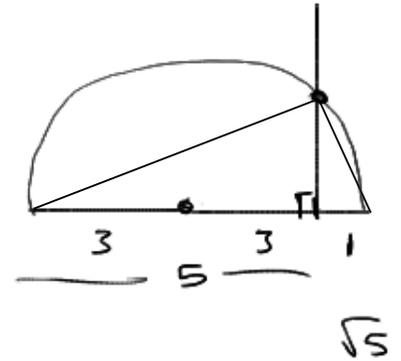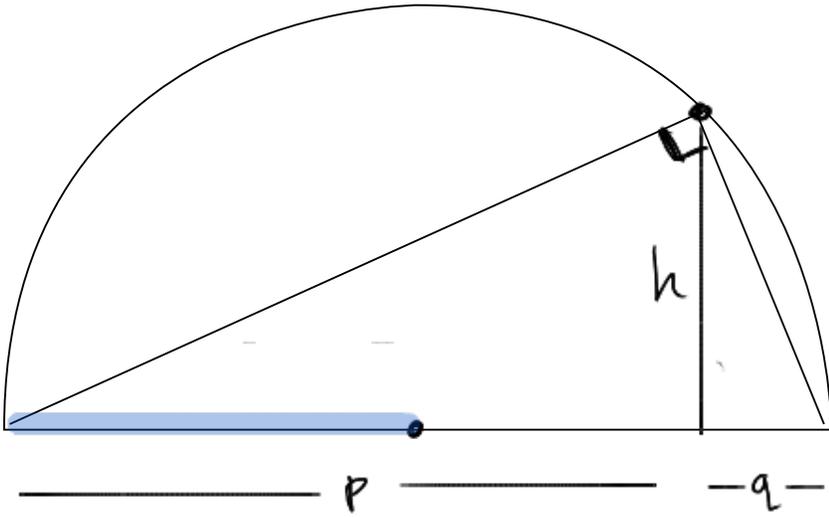


Let $q = 1$, then radius is $\dfrac{p+1}{2}$

$\therefore h = \sqrt{pq}$  so  $h = \sqrt{p}$

And speaking of Thales' theorem, it is a special case of

**The Inscribed Angle Theorem**
   The angle on the boundary of a circle is twice that at the center.

When theta = 90, the angle at the center is 180, so the angle is on a diameter.

Inscribed angle in Desmos

https://www.desmos.com/calculator/8anwclpi4d

Thm: psi = 2 theta

pf: triangle angle sum / supplementary