Other Geometrically Constructible Numbers

1. Definition
   a. Length of constructible segment
   b. Equivalently, x-coordinate of a constructible point
   c. Points are intersections of constructible segments
2. Given segments with length **a,b** construct segments of length:
   a. **ab**
   b. **a / b**
   c. $\sqrt{a}$
3. The problems above require these antique tools
   a. Intercept Theorem (Thales)
   b. Geometric Mean Theorem (Euclid)

$\boxed{\text{Intercept Theorem}}$ - equivalent to the theorem of Similar triangles —.

Note: $\overleftrightarrow{AC} \parallel \overleftrightarrow{DB}$

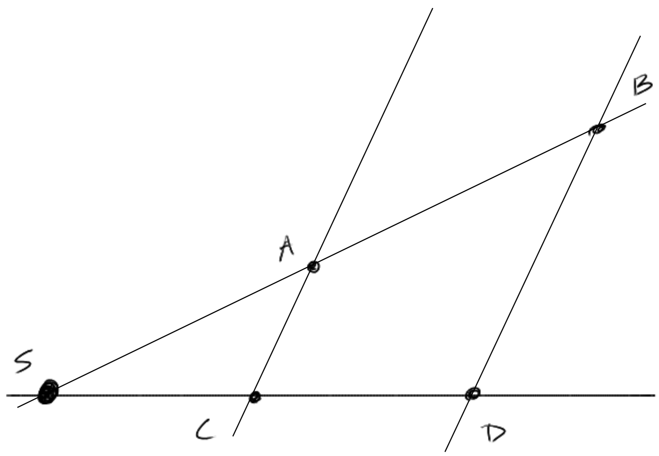
$$\frac{SC}{CD} = \frac{SA}{AB}$$

$$\frac{SC}{SD} = \frac{SA}{SB}$$

$$\frac{SD}{SC} = \frac{SB}{SA}$$

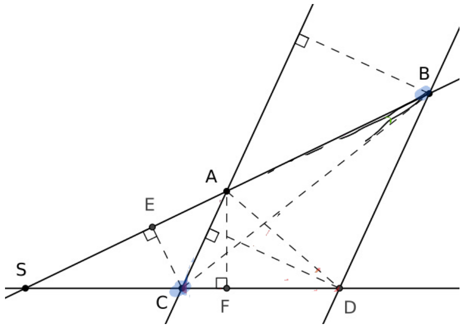
and

$$\frac{SA}{SB} = \frac{AC}{BD}$$



Intercept Theorem

thale's



① Blue: base } two △'s
Green: height

$|\triangle ACD| = |\triangle ABC|$

② $|\triangle CDA| = |\triangle CBA|$
and
$|\triangle SDA| = |\triangle SCB|$

$|\triangle SDA| = |\triangle SCB|$

Notation: For a triangle the vertical bars ($|\ldots|$) denote its area and for a line segment its length.

Proof: Since $CA \parallel BD$, the altitudes of $\triangle CDA$ and $\triangle CBA$ are of equal length. As those triangles share the same baseline, their areas are identical. So we have $|\triangle CDA| = |\triangle CBA|$ and therefore $|\triangle SCB| = |\triangle SDA|$ as well. This yields

$$\frac{|\triangle SCA|}{|\triangle CDA|} = \frac{|\triangle SCA|}{|\triangle CBA|} \text{ and } \frac{|\triangle SCA|}{|\triangle SDA|} = \frac{|\triangle SCA|}{|\triangle SCB|}$$

Plugging in the formula for triangle areas ($\frac{\text{baseline}\cdot\text{altitude}}{2}$) transforms that into

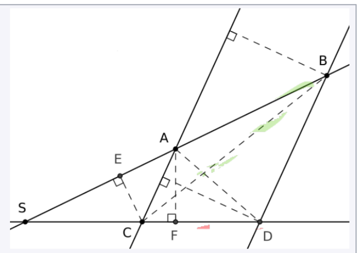
$$\frac{|SC||AF|}{|CD||AF|} = \frac{|SA||EC|}{|AB||EC|} \text{ and } \frac{|SC||AF|}{|SD||AF|} = \frac{|SA||EC|}{|SB||EC|}$$

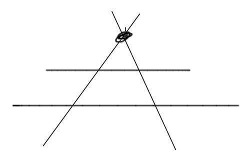
Canceling the common factors results in:

(a) $\frac{|SC|}{|CD|} = \frac{|SA|}{|AB|}$ and (b) $\frac{|SC|}{|SD|} = \frac{|SA|}{|SB|}$

Now use (b) to replace $|SA|$ and $|SC|$ in (a): $\frac{\frac{|SA||SD|}{|SB|}}{|CD|} = \frac{\frac{|SB||SC|}{|SD|}}{|AB|}$

Using (b) again this simplifies to: (c) $\frac{|SD|}{|CD|} = \frac{|SB|}{|AB|}$ □





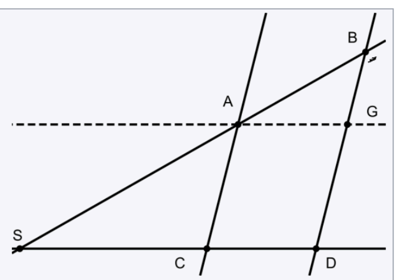
Draw an additional parallel to SD through A. This parallel intersects BD in G.

Then one has $|AC| = |DG|$ and due to claim 1 $\frac{|SA|}{|SB|} = \frac{|DG|}{|BD|}$ and

therefore $\frac{|SA|}{|SB|} = \frac{|AC|}{|BD|}$

□

(✱)



"S" $\frac{SA}{SB} = \frac{SC}{CD}$

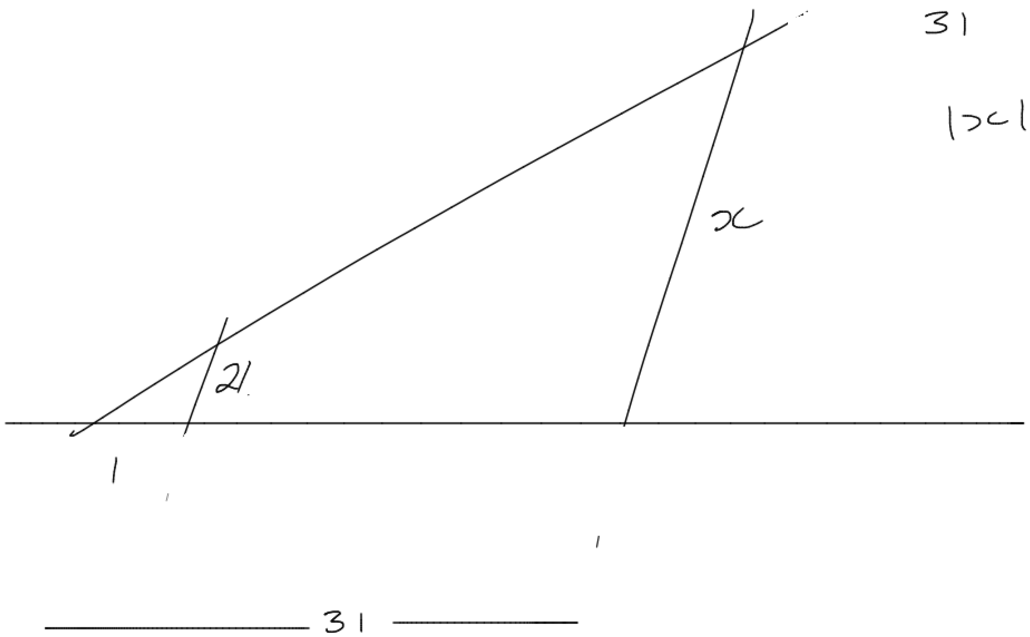
"B" $\frac{AB}{SB} = \frac{BG}{BD}$

$\frac{AB}{SA} = \frac{BG}{GD}$

$\frac{SB}{AS} = \frac{BD}{GD}$ $\Rightarrow \frac{SA}{SB} = \frac{GD}{BD}$ (✱)

Thales' Intercept Theorem and ab and a/b

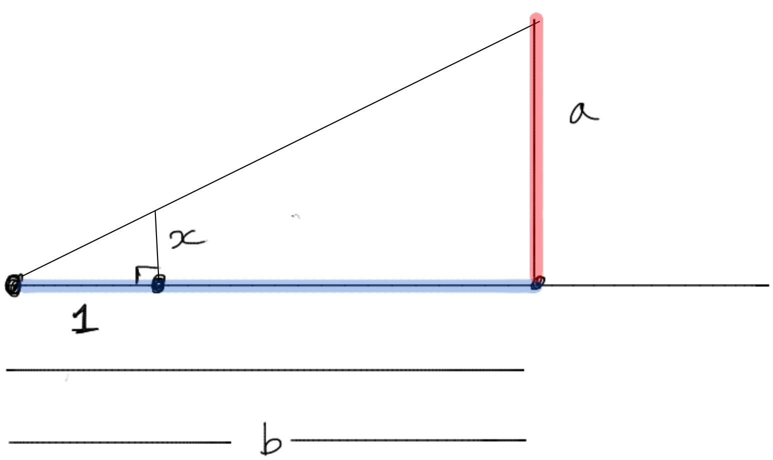
31 & 21 $\Rightarrow$ Construct a segment of length $31 \cdot 21 = 651$



$$\frac{|x|}{31} = \frac{21}{1}$$

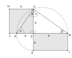
$$|x| = 21 \cdot 31$$

Can you construct a segment of length $\frac{a}{b}$?

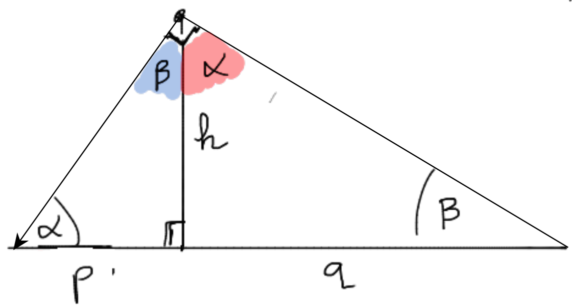


$$\frac{|a|}{|b|} = \frac{|x|}{1}$$

$$|x| = \frac{|a|}{|b|}$$

The geometric mean theorem 

$\left\{ \text{use similar } \triangle\text{'s to show } h = \sqrt{pq} \right\}$



Similar $\triangle$'s

$$\frac{p}{h} = \frac{h}{q}$$

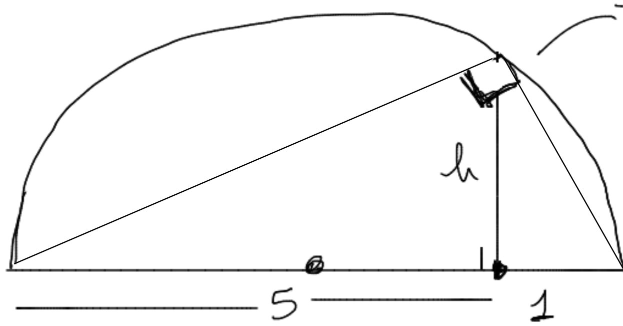
$$h^2 = pq$$

$$h = \sqrt{pq}$$

Seg. of length $\sqrt{5}$

$\sqrt{5} = \sqrt{5 \cdot 1}$, set $p = 5$, $q = 1$

Thale's $\Rightarrow 90^\circ$

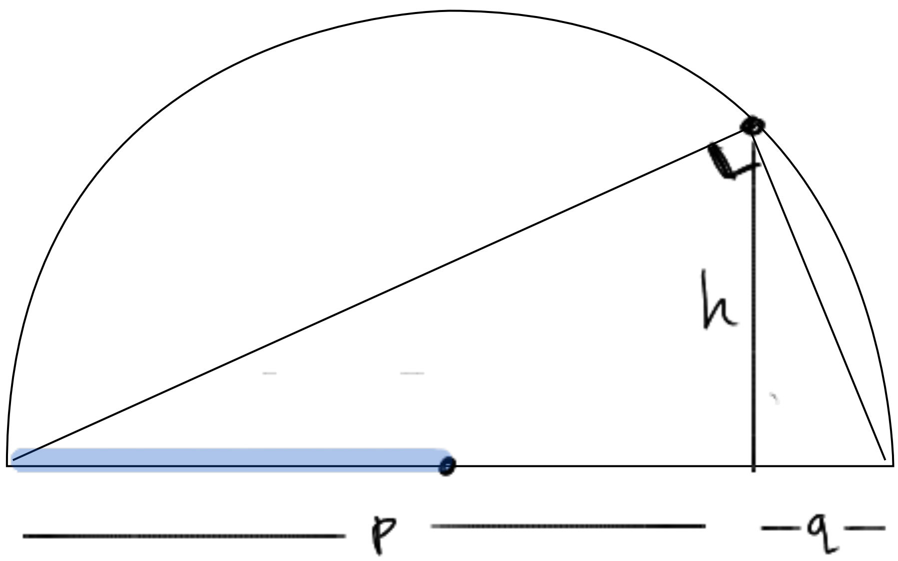


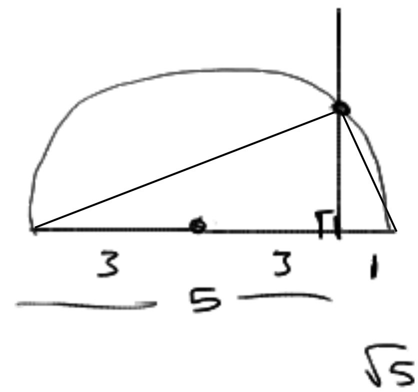
Geom Mean $\Rightarrow$

$h = \sqrt{5 \cdot 1} = \sqrt{5}$

To construct a segment of length 5, draw a circle of radius 3, raise a perpendicular off the diameter at 1 unit away from edge. You raised root 5. yeah.

Use Geometric Mean Theorem to construct $\sqrt{p}$





Let $q = 1$, then radius is $\frac{p+1}{2}$

$\therefore$ $h = \sqrt{pq}$ so $h = \sqrt{p}$

And speaking of Thales' theorem, it is a special case of

The Inscribed Angle Theorem

The angle on the boundary of a circle is twice that at the center.

When theta = 90, the angle at the center is 180, so the angle is on a diameter.

Inscribed angle in Desmos

https://www.desmos.com/calculator/8anwclpi4d

Thm: psi = 2 theta

pf: triangle angle sum / supplementary