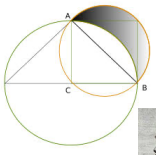
Euclid's Proof of the Pythagorean Theorem

1. Greece flourished in the 150 years between Hippocrates & Euclid.
   a. Plato (427 BC) and his student Eudoxus (408 BC) led the way
      i. Many sophists (clever know-it-alls who taught for money) wandered the land - some brilliant, some quacks
      ii. **Socrates**' disciple was **Plato**, and his student was **Aristotle**
   b. Plato traveled until age 40, then founded the Academy in Athens. The intellectual center of the world, headquarters of Greek wisdom.
      i. Learned from the Pythagoreans in Italy, visited Egypt, etc. Sold as a slave, but then ransomed by his friends.
      ii. His school is the spiritual ancestor of our institution of higher learning, the land had belonged to the hero Akademos.
      iii. As was custom, to establish legitimacy the Academy was a "religious brotherhood" worshiping the Muses
      iv. For 900 years it stood as intellectual center of Greece until 529 AD, was closed b/c it was "pagan" and of "perverse learning"
   c. Math was king, Geometry was required for entrance.
      i. Plato disliked "applied mathematics" and geometric instruments, embracing only the straightedge and compas
   d. **Eudoxus** was poor, commuted in to Athens, and had two main pieces of mathematics, was 2nd only to Archimedes
      i. Theory of proportion
         1. Many proofs of "similar triangles have equal ratios of corresponding sides" required the (false) commensurability of numbers.
         2. Eudoxus' devised a proof that did not appeal to commensurability.
         3. His proof is found in Book V of the Elements (Euclid didn't prove everything there.)
      ii. Method of Exhaustion
         1. Main idea of limits/calculus.
         2. Archimedes used/credited Eudoxus' idea to find the area of a circle.
   e. Alexander the Great & Alexandria
      i. 332 BC Alex (20 y/o from Macedonia) conquered Egypt, established a new city Alexandria
      ii. Alexandria grew while Alex "conquered the world" and became the intellectual center
      iii. 500 K people and 600K (really?) Papyrus rods, Alexandrian Library & Museum > Academy,
      iv. The Museum (the oldest university) was heaven on earth for thinkers, food, space community.
      v. Alex conquered Greece, Egypt, most of Asia Minor, and died trying to capture India.
      vi. Alexandria remained central & powerful for 300 years - the Hellenistic (Greek-like) age, until the Roman Empire.
      vii. 300 - 100 BC is the 2nd most mathematically productive era ever. #1 = Kepler - Gauss (1600 -1850)
      viii. Archimedes, Eratosthenes, Apollonius Pappus, Claudius Ptolemy, Diophantus & Euclid all worked here at some time.

Euclid's Proof of the Pythagorean Theorem

1. Euclid came to Alexandria to establish a school of mathematics (300 BC)
   a. Trained at the Academy, but hereafter Greek mathematics had Alexandrian roots
   b. Founded a school in Alexandria
   c. Two Euclidean stories
      i. King Ptolemy asked about a shorter way to learn geometry than reading The Elements
         1. "There is no royal road to geometry"
      ii. A young student asks Euclid: "But what shall I get by learning these things?"
         1. Euclid points out that knowledge is useful for its own sake then ...
         2. Says to servant, "Give this man a coin, since he must make a profit form what he learns"
            a. This might come from the Pythagorean slogan, "A diagram and a step (in knowledge), not a diagram and a coin."
   d. The Elements of Geometry - only the Bible has been studied more
      i. "The most splendid creation of the Greek mind" - Burton
      ii. 13 books, 435 propositions, 2000+ editions, essential to a liberal education
      iii. plane/solid geometry & number theory
      iv. He organized geometric knowledge beautifully, clearly, all stemming from a few basic assumptions - axiomatically
         1. 5 geometric postulates, 5 common notions, 23 definitions = 435 propositions
         2. the choice of axioms, the arrangement of the propositions, the rigor of demonstration are his own and is amazing
         3. a minimum of assumptions and very little that is superfluous
      v. No circular reasoning - required axioms (assumed without proof)
      vi. When Rome fell, Arab scholars carried it to Baghdad, reappeared in the Renaissance,
      vii. Studied by Newton & Leibniz, Napoleon, Lincoln (40 y/o read to train his logical approach, Bertrand Russell



150 years

Hippocrates

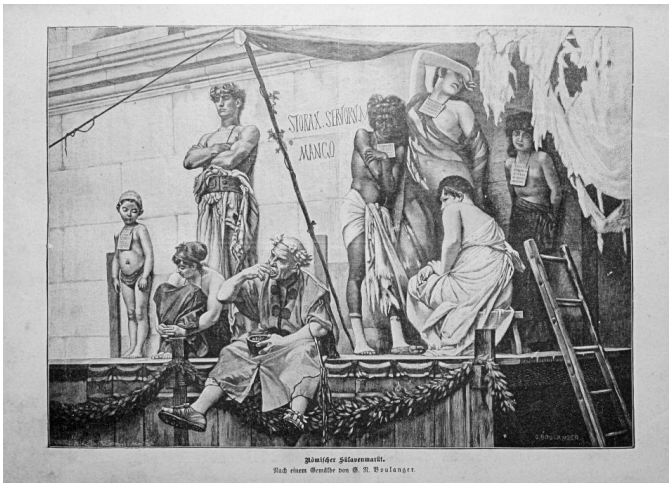
disliked Applied Math

427 BC

Plato - Academy @ Athens
lasted 900 years → until Romans

Eudoxus - 408 BC
1. Exhaustion
2. proportion
proof
↳ appears in Euclid's Elements book V,

Euclid - 300 BC



Römischer Sklavenmarkt.
Nach einem Gemälde von G. A. Boulanger.

Thales:

Knew that similar Δs were proportional.

at the time the proof of this fact rested upon "commensurability of magnitudes"

"the all lengths are commensurate (rationality of the real #s ⟹ similar Δ theorem holds."

Eudoxus

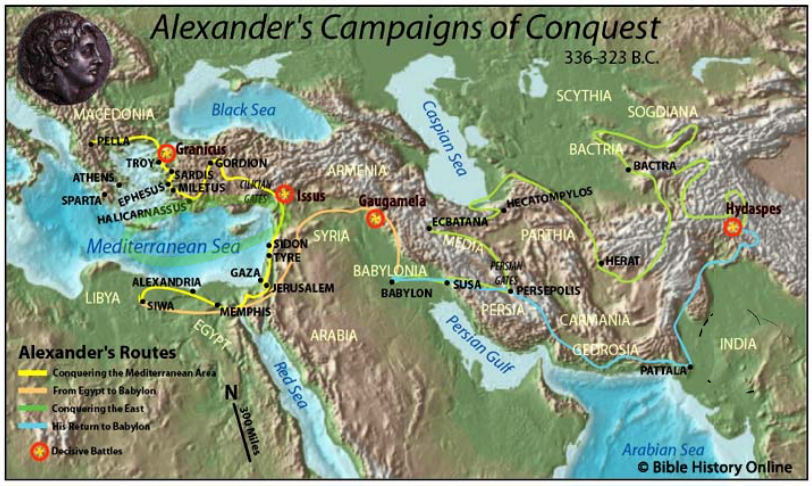
Filled in this gap

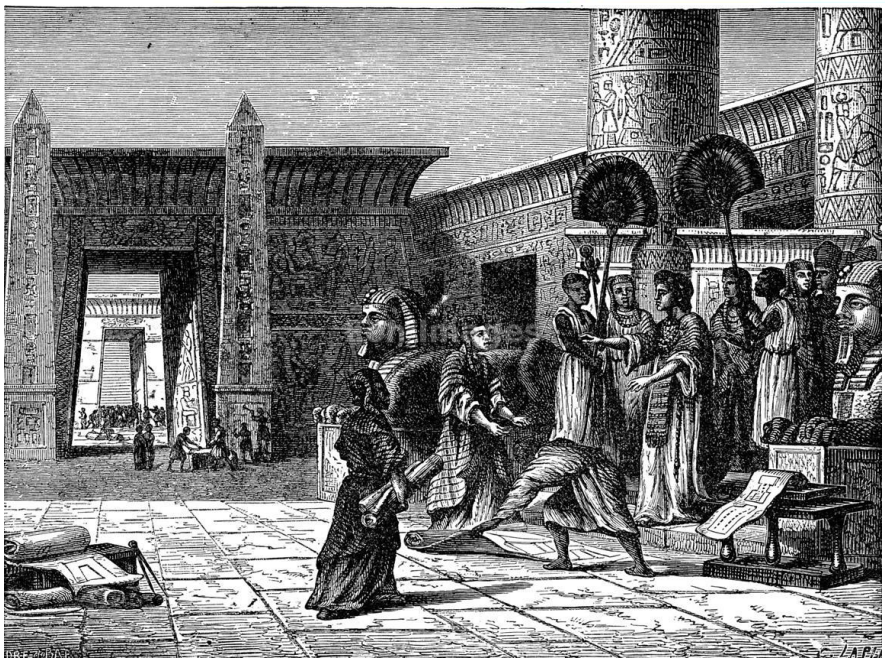
— NO one doubted the similar Δ thm, but when $\sqrt{2} \neq \frac{a}{b}$, this invalidated the proof of Sim. Δ Thm

— Eudoxus proved this w/o using commensurability.

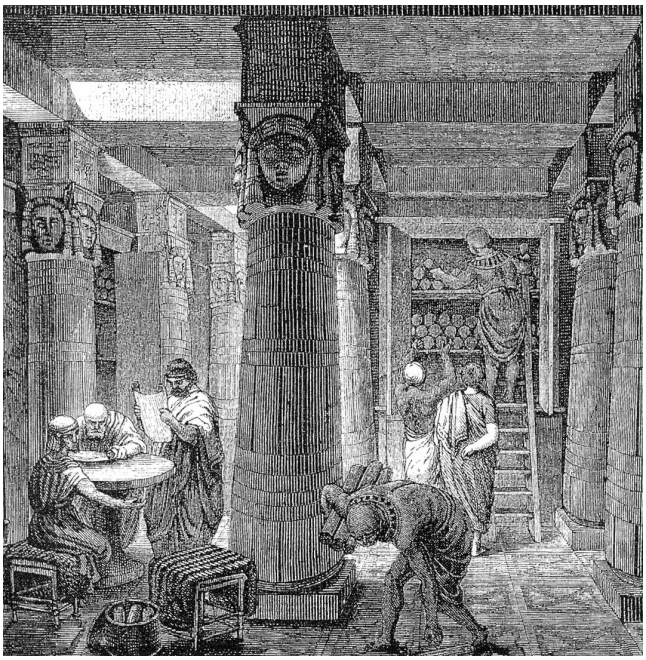
— This proof is found in Book V of the Elements.

Alex the Great 332BC





500K people & 600K Papyrus rods

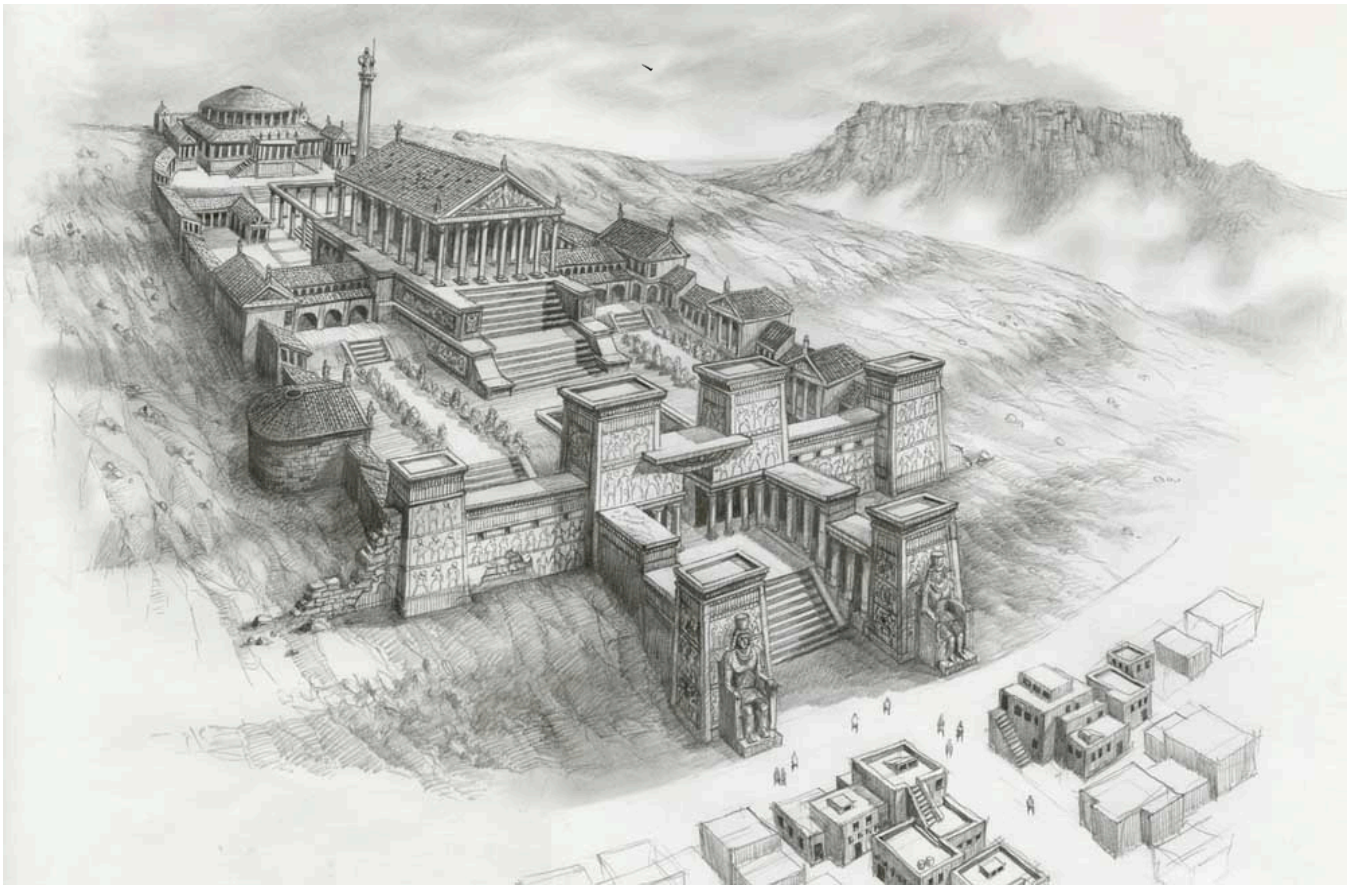


19th century depiction of Library @ Alexandria

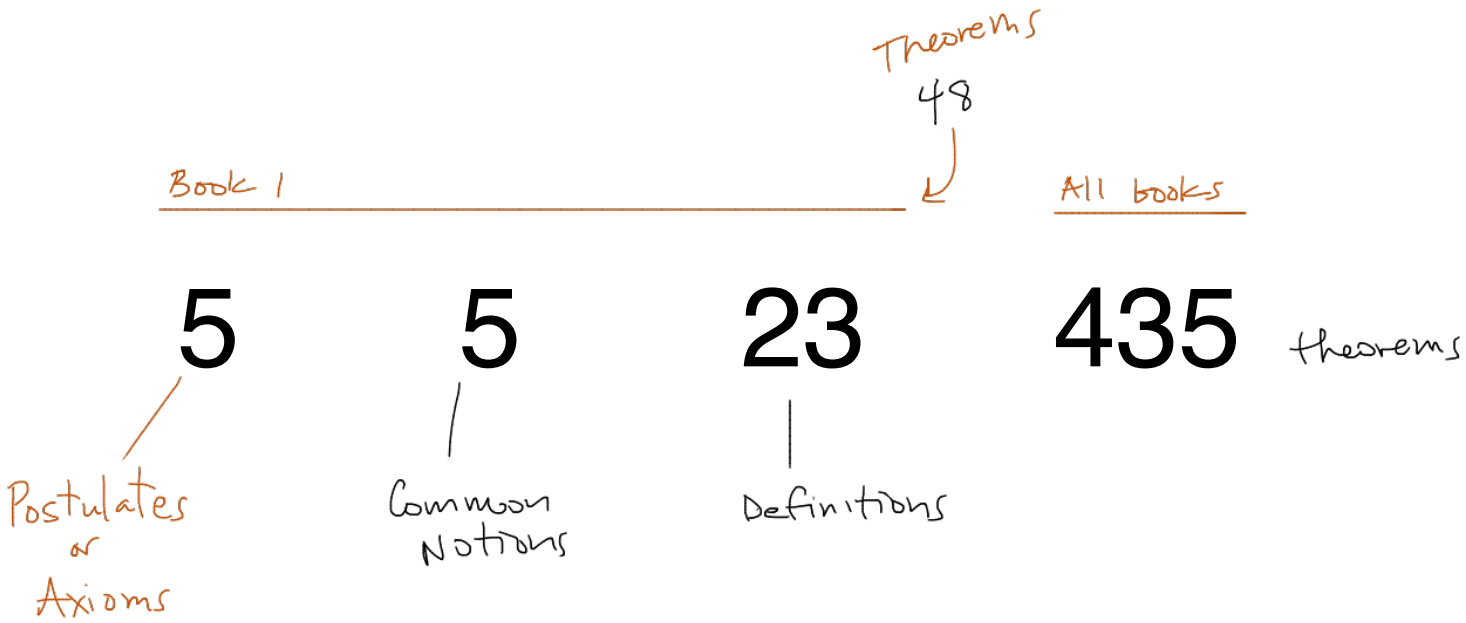
People who studied @ Library of Alexandria

- Archimedes
- Erastosthenes
- Pappus
- Ptolemy
- Diophantus
- Euclid

The Library @ Alexandria Euclid: coin story.



Euclid's Elements : 13 books



Common Notions

Common notion 1.
Things which equal the same thing also equal one another.

Common notion 2.
If equals are added to equals, then the wholes are equal.

Common notion 3.
If equals are subtracted from equals, then the remainders are equal.

Common notion 4.
Things which coincide with one another equal one another.

Common notion 5.
The whole is greater than the part.

Euclid - Book I

1. Preliminaries

a. Got to start some place

i. Point: That which has no part

ii. Line: Breadthless width

iii. Straight line: A line that lies evenly between its points

b. Modern Geometry

i. Point, Line are undefined terms

c. Other definitions

i. Right - when a line stands on another and the adjacent angles are equal

ii. No mention of angles in the Elements

iii. Parallel - when two straight lines do not meet

d. The Postulates

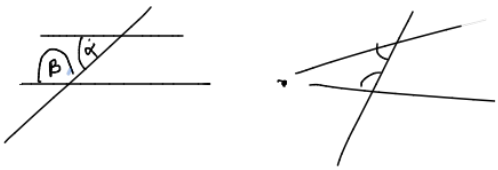
i. Two points determine a line

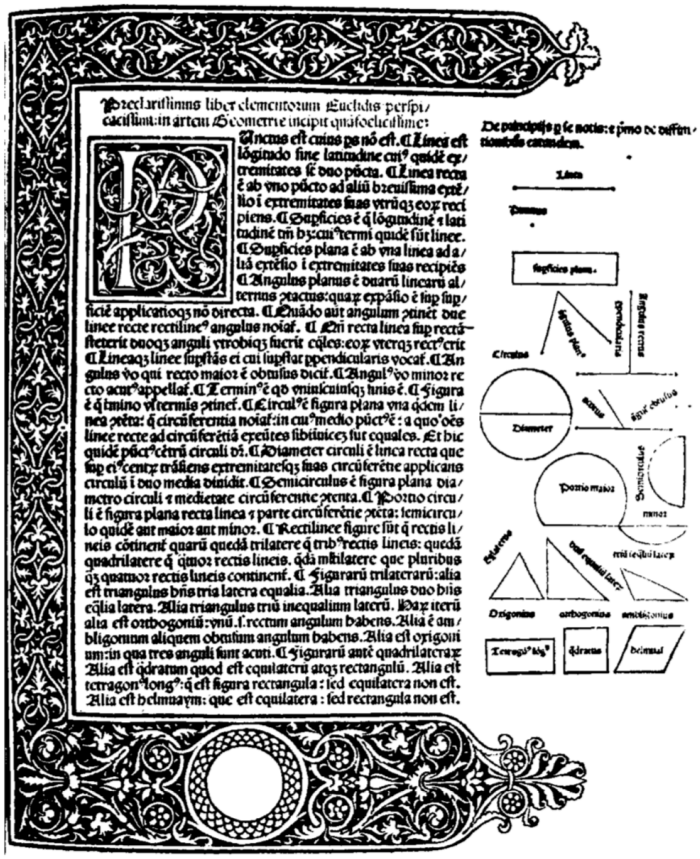
ii. Lines can be extended arbitrarily

iii. A center and distance (radius) determine a circle

iv. All right angles are equal

v. If a line falling on two straight lines make the **interior angles on the same side less than two right angles**, the two straight lines, if produced indefinitely, **meet** on that side on which are the angles less than the two right angles.

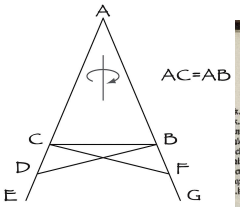


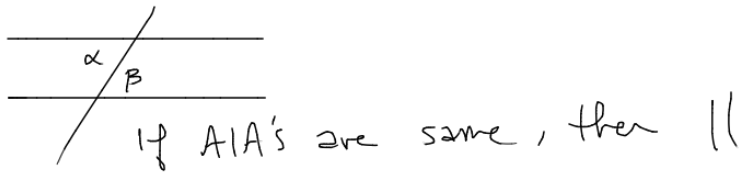


A page from the rst printed edition of Euclid's *Elements*. Published in Latin in 1482. *(Courtesy of Burndy Library.)*

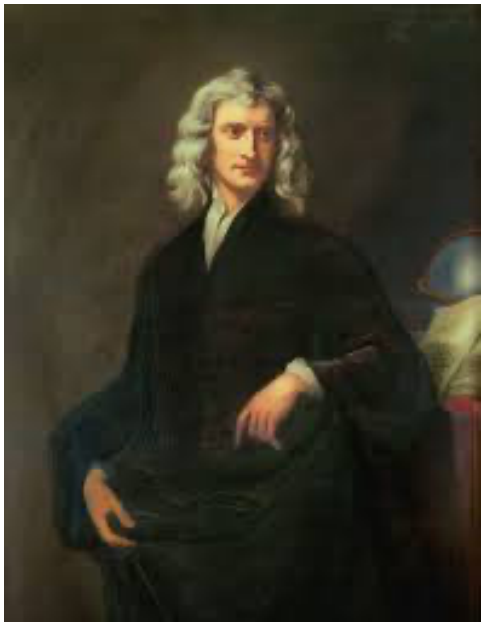
Euclid's Elements - Book I

- **1. Propositions 1 - 26**
  - **a. Triangle Congruence**
    - i. SAS
    - ii. SSS & ASA follow from SAS
  - **b. Constructing perpendiculars**
    - i. It is implied that exactly ONE perp can be constructed
  - **c. Isosceles triangles**
    - i. Base angles are equal
    - ii. The last proposition of Euclid taught in the Middle Ages
    - iii. Flight of the fools
      1. Students abandoned geometry afterward
    - iv. Bridge of Asses
      1. The proof's figure esembled a bridge, and stumped many
      2. Euclid's proof was complicated, Proclus' shorter, Pappus' was flip it "Irish Bull"
- **2. Propositions 27+ : Parallels**
  - **a. AIA congruent implies parallel**
    - i. doesn't use I.V
  - **b. Parallel lines CAN be constructed without I.V**
  - **c. Sum of angles is 2 right angles (180 degrees)**
    - i. requires I.V.
  - **d. Many pieces of geometry rely on I.V.**
    - i. Quadrilaterals





$\alpha + \beta + \gamma = 180^\circ$ uses I.V. } Euclidean Geometry.

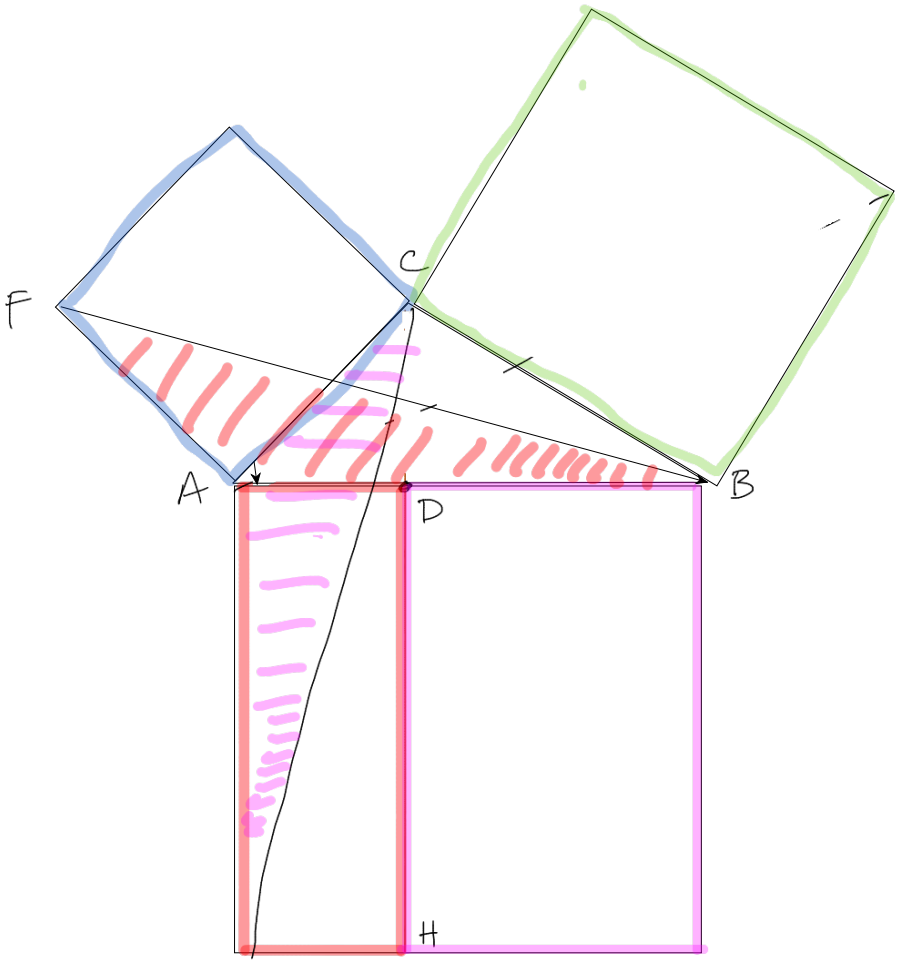


Leibniz

Napolean

Euclid's Proof of the Pythagorean Theorem

show $|AB|^2 = |AC|^2 + |BC|^2$



Idea: Blue Area □
=
Red Rectangle

Green □ = Pink □