Monday Week 6

1. Final Project
    a. Producible: open
        i. paper
        ii. website
        iii. poster
2. Today
    a. Finish Infinitude of Primes
    b. Golden Ratio

Number Theory in the Elements

1. Book VII - IX (without equations)
2. Euclidean Algorithm
   a. Find the GCD of two numbers
   b. It shows the GCD of two relatively prime numbers is 1
3. VII.30: prime p, plab implies pla or plb
4. VII.31: any composite is divisible by some prime
   a. key: well-ordering principle (positive ints)
5. IX.14: **PROPOSITION IX.14** If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.
   a. Unique Factorization / Fund. Thm. Arithmetic
   b. Application: 8^N doesn't end in 0 for any N
      i. (if it did, it'd be divisible by 5, but 8^N is just 2's)
6. Towards infinitude:
   a. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151
      i. the first 36, 25 of which are less than 100
   b. sparsity: between 10,000,001 and 10,000,100 there are only 2

Euclidean Algorithm:

(14, 64) — what's the gcd

$64 = 14 \cdot 4 + 8$

$14 = 8 \cdot 1 + 6$

$8 = 6 \cdot 1 + 2$

$6 = 2 \cdot 3 + 0$

$2 = \gcd$

(108, 3)

$108 = 3 \cdot 36$

(108, 5)

$108 = 5 \cdot 21 + 3$

$5 = 3 \cdot 1 + 2$

$3 = 2 \cdot 1 + 1$ → Relatively Prime

$2 = 1 \cdot 1 + 1$

★

BTW: Rel prime $\Rightarrow \exists\, a, b \in \mathbb{Z}$ s.t.

$108(a) + 5(b) = 1$

$108(2) + 5(-43) = 1$

3. prime p, $\underbrace{p \mid ab}$ $\Rightarrow$ $p \mid a$ or $p \mid b$

(p divides ab means $ab = mp$

proof: Assume $p \mid ab$.

If $p \mid a$ we're done.

If $p \nmid a$ $\Rightarrow \gcd(p, a) = 1$ so $pm + an = 1$ (see ★)

(not divides

Euclid's Lemma: $a \mid bc$ & $\gcd(a, b) = 1$ then $a \mid c$

By Euclid's lemma $p \mid ab$ & $\gcd(p, a) = 1$ so $p \mid b$.

(Euclid's lemma holds b/c:

$\Rightarrow 1 = px + ay$ from

$\Rightarrow b = pbx + a\underbrace{by}$ multiply by b

by assumption is a mult of p

$\Rightarrow$

$$\underbrace{b = pbx + psy = p(bx + sy)}$$

b is a multiple of p $\therefore$ $p \mid b$.

Unique Factorization: Any whole number decomposes into a product of primes in exactly one way

So, if $n \in \mathbb{Z}$, $n = p_1 \cdot \ldots \cdot p_k$ & p_i are unique.

Proposition: there are infinite # of primes.

- This is due to Euclid
- This is perhaps surprising.

proof: By contradiction, suppose $\{2, 3, 5, 7, \ldots, P_k\}$ are all the primes.

Form $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot P_k + 1$.

this factors uniquely into a $\underbrace{\text{product of primes}}_{\text{these are the ones in list}}$, too.

So there is some P_i in my list that divides N

$N = P_i \cdot P_1 \cdot P_2 \cdots P_\ell$, but P_i also divides $2 \cdot 3 \cdot 5 \cdot \ldots \cdot P_k$

$\Rightarrow$ $p_i \mid N$ and $p_i \mid N-1$

$\Rightarrow$ if p divides a & b then $p \mid a-b$

$\Rightarrow$ $p_i \mid 1$ $\otimes$ (all primes > 1 $1 \neq P_i \cdot m$)

Two kinds of primes, greater than 2.

$p = 17 = 4 \cdot 4 + 1$

BLUE

$5 = 4 \cdot 1 + 1$

$13 = 4 \cdot 3 + 1$
$= 4 \cdot 4 - 3$

$4k+1$

$p = 31 = 4 \cdot 7 + 3$

MAGENTA

$4k+3$

$11 = 4 \cdot 2 + 3$
$= 4(3) - 1$

$19 = 4 \cdot 5 - 1$

there are an infinite # of both kinds of primes

(Hint:

EX $\{7, 3\} \Longrightarrow 7^2 = 49 = 4 \cdot 8 + 1$

magenta   blue

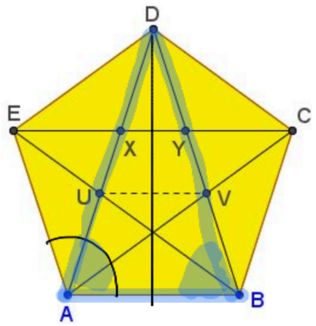
EX $\{5, 13\} \Rightarrow 5^2 = 25 = 4 \cdot 6 + 1$

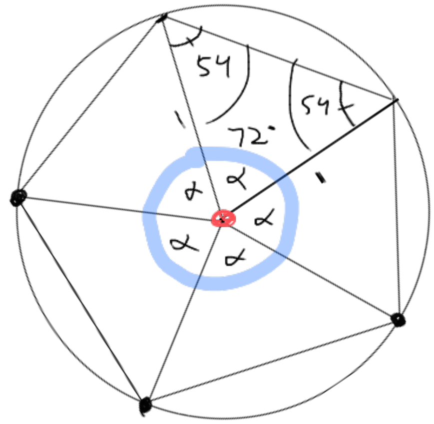
blue   blue

Golden Ratio in Regular Pentagon

$72 + 2\beta = 180$

The *golden ratio*, $\phi = \dfrac{1+\sqrt{5}}{2}$, makes frequent and often unexpected ***appearance in geometry***. Regular pentagon - the *pentagram* - is one of the places where the golden ratio appears in abundance.





To mention a few (some of which have been proved ***elsewhere***, others are straightforward):

$$\frac{DE}{EX} = \frac{EX}{XY} = \frac{UV}{XY} = \frac{EY}{EX} = \frac{BE}{AE} = \phi.$$

$5\alpha = 360 = 2\pi$

$\alpha = \dfrac{2\pi}{5} = 72^\circ$

$\cos\left(\dfrac{2\pi}{5}\right) = 1.618\ldots$ golden ratio