#7 Show $x = \sqrt[3]{3} + \sqrt{2}$ is algebraic.

what does this mean?
("$\sqrt[3]{2} + \sqrt{2}$" is a zero of some polynomial with integer coeffs)

(idea: try an easier one)

$x = \sqrt{2}$ is algebraic

$x^2 = 2$

$x^2 - 2 = 0$

similarly:

$(x - \sqrt{2}) = \sqrt[3]{3}$

$(x - \sqrt{2})^3 = 3$

(expand.

$x^3 - 3x^2 \cdot \sqrt{2} + 3x \cdot 2 - 2^{3/2} = 3$

$(\sqrt{2})^3 = (2^{1/2})^3$

(I)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

coeffs: stop @ 3

(II) degree sum of each term is 3

(III) signs alternate.

$(a-b)^3 = a^3 - a^2b + ab^2 - b^3$

$x^3 - 3x^2 \cdot \sqrt{2} + 3x \cdot 2 - 2^{3/2} = 3$

separate left/right into integer / non-int:

$x^3 + 6x - 3 = 3\sqrt{2}x^2 + 2\sqrt{2} = \sqrt{2}(3x^2 + 2)$

$x^3 + 6x - 3 = \sqrt{2} \cdot (3x^2 + 2)$

now square both sides

$(x^3 - 6x - 3)^2 = 2(3x^2 + 2)^2$

you can stop here, stating from this we can expand getting a polynomial in standard form

(b) $x = \dfrac{1}{\sqrt[3]{3} + \sqrt{2}}$

$x(\sqrt[3]{3} + \sqrt{2}) = 1$

distribute & expand..

(#10

$\sqrt[3]{3} \cdot x + \sqrt{2}x = 1$

$\sqrt[3]{3}x = 1 - \sqrt{2}x$

$3x^3 = (1 - \sqrt{2}x)^3$

similar to before.

7 (b)

$$x = \frac{1}{\sqrt[3]{5} + \sqrt{7}} \qquad \text{what poly is this a root of ?}$$

$$x(\sqrt[3]{5} + \sqrt{7}) \overset{①}{=} 1$$

② ||

$$x^3(\sqrt[3]{5} + \sqrt{7})^3 \overset{③}{=} x^3(5 + 3\cdot 5^{2/3}\sqrt{7} + 3\cdot 5^{1/3}\cdot 7 + 7\sqrt{7}) = 1$$

$$\overset{④}{=} 5x^3 + 3x^3\cdot 5^{2/3}\cdot 7^{1/2} + 21x^3\cdot 5^{1/3} + 7^{3/2}x^3 = 1$$

⑤ $$3x^3\cdot 5^{2/3}\cdot 7^{1/2} + 7^{3/2}x^3 = 1 - 5x^3 - 21x^3\cdot 5^{1/3}$$

⑥ $$\sqrt{7}(3x^3\cdot 5^{2/3} + 49x^3) = \ldots$$

⑦ $$7(3x^3\cdot 5^{2/3} + 49x^3)^2 = (1 - 5x^3 - 21x^3\cdot 5^{1/3})^2$$

$$7(9x^6\cdot 5^{4/3} + 6x^3\cdot 5^{2/3}\cdot 49x^3 + 49^2x^6) = (1-5x^3)^2 - 42(1-5x^3)(x^3\cdot 5^{1/3}) + 21^2x^6\cdot 5^{2/3}$$

$5^2\cdot 7^3\cdot 6$

⑧ $$5^{1/3}[63x^6 + 5^2\cdot 6\cdot 7^3x^3 + 42(1-5x^3)x^3 - 21^2x^6\cdot 5^2] = 1 - 10x^3 + (25-7^3)x^6$$

cube

⑨ $$5[63x^6 + 5^2\cdot 6\cdot 7^3x^3 + 42x^3(1-5x^3) - 21^2\cdot 5^2\cdot x^6]^3 = (1 - 10x^3 + (25-7^3)x^6)^3$$

7 (b)

$$x = \frac{1}{\sqrt[3]{5} + \sqrt{7}}$$ what poly is this a root of?

$$x\left(\sqrt[3]{5} + \sqrt{7}\right) = 1$$ cross mult.

$$5^{1/3}x + 7^{1/2}x = 1$$ distribute

$$5^{1/3}x = 1 - 7^{1/2}x$$ separate

$$5x^3 = \left(1 - 7^{1/2}x\right)^3$$ cule

$$= 1 + 3\cdot 7\, x^2 - 3\cdot 7^{1/2} + 7^{3/2} x^3$$ expand

$$5x^3 - 21x^2 - 1 = 7^{1/2}\left(7x^3 - 3\right)$$ collect (radicals on one side)

square (remove radical)

$$\left(5x^3 - 21x^2 - 1\right)^2 = 7\left(7x^3 - 3\right)^2$$

from here expand / collect

$x = \sqrt[3]{2} - \sqrt{3}$

Similar to # 7

$(x + \sqrt{3}) = \sqrt[3]{2}$

$x^3 + 3x^2\sqrt{3} + 3x \cdot 3 + 3\sqrt{3} = 2$

$x^3 + 9x - 2 = -3x^2\sqrt{3} - 3\sqrt{3}$

$= -3\sqrt{3}(x^2+1)$

$((x^3+9x)-2)^2 = 27(x^2+1)^2$

$(x^3+9x)^2 - 2\cdot2(x^3+9x) + 4 = 27(x^4+2x^2+1)$

- $x^6 + 18x^4 + 81x^2 - 4x^3 - 36x + 4 = 27x^4 + 54x^2 + 27$

$P(x) = x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$

note: $p(\sqrt[3]{2} - \sqrt{3}) = 0$

$x = \dfrac{1}{\sqrt[3]{2} - \sqrt{3}}$

$x(\sqrt[3]{2} - \sqrt{3}) = 1$

$x \cdot \sqrt[3]{2} = x\sqrt{3} + 1$

$2x^3 = 3x^3\sqrt{3} + 9x^2 + 3x\sqrt{3} + 1$

$2x^3 - 9x^2 - 1 = 3x\sqrt{3}(x^2+1)$

$(2x^3 - 9x^2 - 1)^2 = 9x^2 \cdot 3 \cdot (x^2+1)^2$

$(2x^3-9x^2)^2 - 2(2x^3-9x^2) + 1 = 27x^2(x^4 + 2x^2 + 1)$

$4x^6 - 36x^5 + 81x^4 - 4x^3 + 18x^2 + 1 = 27x^6 + 54x^4 + 27x^2$

$P(x) = 23x^6 + 36x^5 - 27x^4 + 4x^3 + 9x^2 - 1$

kills 3

$f(x) = x^2 - 7x + 12$

$f(3) = 0$

kills $\sqrt{3}$

$g(x) = f(x^2) = x^4 - 7x^2 + 12$

$g(\sqrt{3}) = 0$

(if you square the input, then the new poly evaluates to the old poly @ the square-root of the input)

kills $\sqrt[3]{3}$

$\sqrt[3]{3} \Leftrightarrow f(x^3) = x^6 - 7x^3 + 12$

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double input, vanishes on 1/2 vanishing set of original

$h\left(\frac{3}{2}\right) = 0 \Leftrightarrow f(2x) = (2x)^2 - 7(2x) + 12$

$h(x) = 4x^2 - 14x + 12$

$h(1.5) = f(3) = 0$

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want

$k(4) = 0$

Take a poly that vanishes @ $x=3$ & decrease input by 1

$f(x-1)$

$= (x-1)^2 - 7(x-1) + 12$

$= x^2 - 2x + 1 - 7x + 7 + 12$

$k(x) = x^2 - 9x + 20 = (x-5)(x-4)$

$k(4) = 16 - 36 + 20 = 0$

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properties of algebraics?

$f(a \cdot b) = 0$

$\Rightarrow a \cdot b$ algebraic

$\frac{1}{\pi} \cdot \pi = 1$

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use base poly to get new poly w/ prescribed zeros

$(x-a)(x-b)(x-c) \longrightarrow x^3 + Ax^2 + Bx + C$

or

$P(x) = x^2 - 7x + 12$

kills $x = 3, 4$

want poly that $\sqrt[5]{3}$

$P(x^5) = (x^5)^2 - 7x^5 + 12$

$g(x) = x^{10} - 7x^5 + 12$

$g(\sqrt[5]{3}) = 3^2 - 7 \cdot 3 + 12 = 0$

#9/ c algebraic $\Longrightarrow$ $\frac{1}{2}c$ is too.

Ex: $P(x) = x+1$.
set $P(x)=0$
solve
$\boxed{\begin{matrix} x=-1 \\ \text{root} \end{matrix}}$

Find poly s.t. $\frac{1}{2}(-1)$ is a root
$\quad\quad\quad\quad\quad\quad\quad\quad\; = -\frac{1}{2}$

$$Q(x) = P(2x) = 2x+1$$

kills $-\frac{1}{2}$!

c algebraic $\Longrightarrow$

there is some poly:

$$P(x) = \sum_{i=1}^{n} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

we know

$$P(c) = \sum_{i=1}^{n} a_i (c)^i = 0$$

$$Q(x) = P(2x) = \left\{\sum_{i=1}^{n} a_i (2x)^i\right\} = \sum_{i=1}^{n} a_i 2^i x^i = \sum_{i=1}^{n} 2^i a_i x^i$$

this kills $\frac{1}{2}c$ b/c __________

$$Q\left(\frac{1}{2}c\right) = \sum_{i=1}^{n} a_i \left(2\left(\tfrac{1}{2}c\right)\right)^i = \underbrace{\sum_{i=1}^{n} a_i c^i}_{=0} =$$

11/ c alg $\Rightarrow$ $1+c$

- $P(x) = x^2$, has $x=0$ as a root
- $Q(x) = P(x-1)$

10/ c algebraic $\Longrightarrow$ $\frac{1}{c}$ algebraic

The idea: $x^2-7x+12$ has $x=3,4$ as roots

So $x=3$ is algebraic. So

$$P(3) = 3^2 - 7\cdot 3 + 12 = 0$$

mult. by $\left(\frac{1}{3}\right)^{2 \leftarrow \text{degree}}$ $\Big\}$ $\left(\frac{1}{3}\right)^2\left[3^2-7\cdot 3+12\right] = \left(\frac{1}{3}\right)^2 \cdot 0$

c

$$1 - 7\left(\tfrac{1}{3}\right) + 12\left(\tfrac{1}{3}\right)^2 = 0$$

A poly that kills $\frac{1}{3}$ is $1-7x+12x^2$

Try to make this general

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c alg. $\Rightarrow$ $\sum_{i=1}^{n} a_i x^i$ is a poly & $\sum_{i=1}^{n} a_i c^i = 0$

$x^2-7x+12$ $\qquad$ $3^2-7\cdot 3+12=0$

Now mult. by $\left(\frac{1}{c}\right)^n$

constant, pass across Σ, get new term

#10/ If c is algebraic then $1/c$ is algebraic too $(c \neq 0)$

since #7 relates, let's examine it. ✓

$\Rightarrow$ c is zero of poly. $P(x)$

$\Rightarrow$ There is some $P(x) = \sum_{k=0}^{n} a_k x^k$ for which

eg. $c = 2$.

$P(x) = x^2 - 5x + 6$

$P(2) = 2^2 - 5(2) + 6 = 0$

$$P(c) = \sum_{k=0}^{n} a_k c^k = 0$$

Q: Can I build new poly: $Q(x)$ from $P(x)$ s.t. $Q(1/2) = 0$.

$$\left(\frac{1}{2^2}\right)\left(2^2 - 5(2) + 6\right) = (0)\left(\frac{1}{2^2}\right)$$

$= \left(\frac{1}{2}\right)^{2}$ degree

because I can & it works

$$0 = \frac{2^2}{2^2} - \frac{5 \cdot 2}{2^2} + \frac{6}{2^2} = 1 - 5\left(\frac{1}{2}\right) + 6\left(\frac{1}{2^2}\right)$$

$$0 = 1 - 5\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2$$

this eq'n suggests the answer!

$Q(x) = 1 - 5(x) + 6(x)^2$ or $\boxed{6x^2 - 5x + 1 \Rightarrow Q(x)}$

In general gory detail:

assumption of c being algebraic implies:

$(5x)\lim_{h \to 0} f(h) = \lim_{h \to 0} 5x \cdot f(h)$

$$P(c) = \sum_{k=0}^{n} a_k c^k = 0$$

n — degree

k varies, n is constant

mult by $(c^{-1})^n$

$\left(\frac{1}{c}\right)^n = (c^{-1})^n = c^{-n}$

$$c^{-n} \sum_{k=0}^{n} a_k c^k = 0 \cdot c^{-n} = 0$$

$$\| $$

$$\sum_{k=0}^{n} a_k c^k \cdot c^{-n} = \sum_{k=0}^{n} a_k c^{k-n}$$

$$= \sum_{k=0}^{n} a_k c^{-1(n-k)}$$

$c^{-1(n-k)} =$

$c^{a \cdot b} = (c^a)^b$

$$= \sum_{k=0}^{n} a_k (c^{-1})^{n-k}$$

suggests what the polynomial should be that kills $\frac{1}{c}$

$$= \sum_{k=0}^{n} a_k \left(\frac{1}{c}\right)^{n-k}$$

Transcendental #'s : Def'n : not algebraic

: not the solution to

$$\sum_{k=0}^{n} a_k x^n = 0$$

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$a_i \in \mathbb{Z}$, integers

#12

Suppose $\pi + 1$ is algebraic.

In #11 we showed c being algebraic implied $c+1$ is.

↳ this technique can be used to show $c + r$ is algebraic for any $r \in \mathbb{K}$.

By #11 π must be algebraic since π is an additive integer diff than $\pi + 1$.

#11 c is algebraic.

$1+c$ is too

Ex $f(x) = x^2 - 2x + 1$ $(1,0)$

$g(x) = f(x-4) = (x-4)^2 - 2(x-4) + 1$ $(5,0)$

$= x^2 - 8x + 16 - 2x + 8 + 1$

$g(x) = x^2 - 10x + 25$

$g(5) = 25 - 50 + 25 = 0$

Show

$13 + c$ is algebraic

#15/ Prop. 1.29 is the first two use Parallel Postulate
Prop. 1.31 doesn't. Why didn't Euclid move it in front of 1.29?

1. 1.31 uses 1.27, and 1.28/1.29 make more sense to come immediately after 1.27 than 1.31 — because 1.28 is essentially the same & 1.29 contains converses to 1.28/1.29.

Proclus rightly remarks (p. 376, 14—20) that, as it is implied in I. 12 that only one perpendicular can be drawn to a straight line from an external point, so here it is implied that only one straight line can be drawn through a point parallel to a given straight line. The construction, be it observed, depends only upon I. 27, and might therefore have come directly after that proposition. Why then did Euclid postpone it until after I. 29 and I. 30? Presumably because he considered it necessary, before giving the construction, to place beyond all doubt the fact that only one such parallel can be drawn. Proclus infers this fact from I. 30; for, he says, if two straight lines could be drawn through one and the same point parallel to the same straight line, the two straight lines would be *parallel*, though intersecting at the given point: which is impossible. I think it is a fair inference that Euclid would have considered it necessary to justify the assumption that only one parallel can be drawn by some such argument, and that he deliberately determined that his own assumption was more appropriate to be made the subject of a Postulate than the assumption of the uniqueness of the parallel.

it is implied!

1.12 : uniqueness of ⊥,

1.31 : also implied that parallel's be unique

why delay & not immediatly follow 1.27?

- First needed to "imply" that only one parallel exists.

Prop. 30 implies uniqueness of parallels (because it uses Prop. 29, and thus I.V.)

Euclid wanted to justify the assumption that parallels were unique, first.

- Before constructing the parallel, he wanted to make sure only 1 could be drawn.
- So he did so. But doing so requires I.V. because w/o it multiple exists (2000 years later).

#27 = could have been used as proof of I.47

① #-20 - uses SAS, which comes before Pyth. thm proof. So this ok.

② #21 - supplements of right angles are right - this is the definition of a right angle - ok

but this one requires the area of a trapezoid (see below)`

③ #22, clearly similar Δ's in (a)

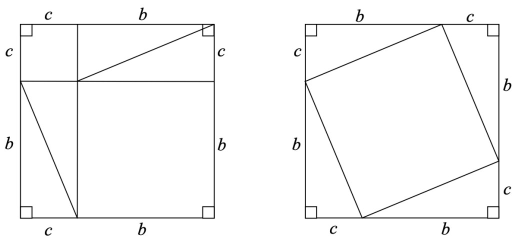
④ #23 similarity

⑤ #24 similar triangles

⑥ #25 book IV about circles

⑦ #26 - uses SAS, (this "could" have been used by Euclid, but the proof of the area of the trapezoid would have had to be a part of it, making somewhat awkward.)

20. The proof that many attribute to Pythagoras himself (i.e., the "ox-killer proof"):



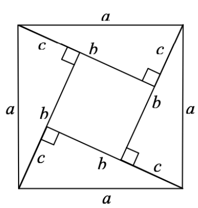
(i) LHS = $c^2 + b^2 + 2bc$

(ii) SAS $\Rightarrow$ hypotenuses are same ... $= a$.

(iii) RHS = $a^2 + 2bc$

(iv) Since LHS = RHS, $a^2 + 2bc = b^2 + c^2 + 2bc$ so $a^2 = b^2 + c^2$.

21. This proof is due to the 12th century Hindu mathematician Bhaskara:



R = right angle

(i) $\angle ca + \angle ab + R = 2R$

$\Rightarrow \angle ca + \angle ab = R$

each corner of large quad. is $\angle ca + \angle ab$

so it's a square (side $= a$)

(ii) supplement of right angle is right so

inside angles are right.

and sides are $b - c$ $\therefore$ square

(iii) $a^2 = (b-c)^2 + 2bc$

$= b^2 - 2bc + c^2 + 2bc$

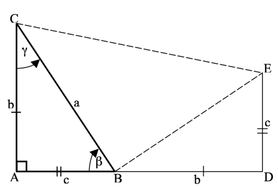
$= b^2 + c^2$

Link to MAA article on Garfield

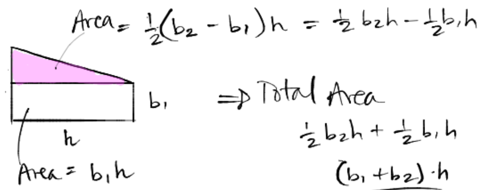
26. The last proof here is due to Congressman (later President) James A. Garfield of Ohio, who published it in the *New England Journal of Education* in 1876.

(a) As a preliminary, prove that the area of a trapezoid is half the product of the height and the sum of the bases.

(b) Now consider right ΔBAC. Extend AB to D so that BD = b and construct DE ⊥ AD at D, with DE = c. Draw BE and CE. With this out of the way, prove that ΔBAC ≅ ΔEDB.



(a)



$\Rightarrow$ Total Area

$\frac{1}{2}b_2h + \frac{1}{2}b_1h$

$\frac{(b_1 + b_2)\cdot h}{2}$

(b) $2R = \beta + \theta + \gamma$ since SAS $\Rightarrow \angle EBD = \gamma$

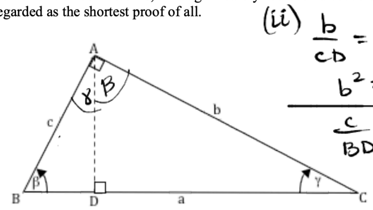
& I.32 $\Rightarrow R + \beta + \gamma = 2R$ so $\beta + \gamma = R$ so $\theta = R$

(c) $\frac{(b+c)(c+b)}{2} = \frac{1}{2}bc + \frac{1}{2}a^2 + \frac{1}{2}bc$ } $b^2 + 2bc + c^2 = bc + a^2 + bc$

so $a^2 = b^2 + c^2$

(area of trap formula)

22. This proof of the Pythagorean Theorem is usually credited to the 17th century British mathematician John Wallis, although it surely had been discovered prior to him. It is regarded as the shortest proof of all.



(ii) $\frac{b}{CD} = \frac{a}{b}$

$b^2 = a \cdot CD$

$\frac{c}{BD} = \frac{a}{c}$

$c^2 = a \cdot BD$

$a \cdot CD + a \cdot BD$

$= a(a - BD) + a \cdot BD = a^2$

$a^2 = b^2 + c^2$

(i) ΔABD: $\beta + R + \theta = 2R$

so $\beta + \theta = R$

$\theta = R - \beta$

ΔACD: $\gamma + R + \delta = 2R$

$\delta = R - \gamma$

ΔABC: $\beta + \gamma + R = 2R$

$\beta = R - \gamma$

$\theta = \delta$