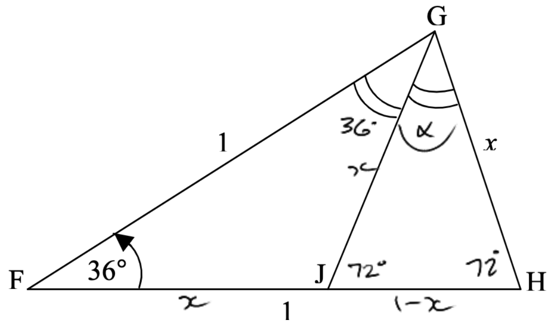


$\angle FGJ = 36°$ by Pons As,

$\angle FJG = 108°$ by $\Delta\Sigma = 180$

$\angle GJH = 72°$

ΔFGH isosceles since $\overline{FG} = \overline{FH}$

$\Rightarrow 36 + \alpha = \angle GHJ$ and

$36 + \alpha + \angle GHJ + 36 = 180$ so

$72 + 2\alpha + 36 = 180 \Rightarrow$

$\alpha = 36$

then $36 + 72 + \angle GHJ = 180 \Rightarrow \angle GHJ = 72°$

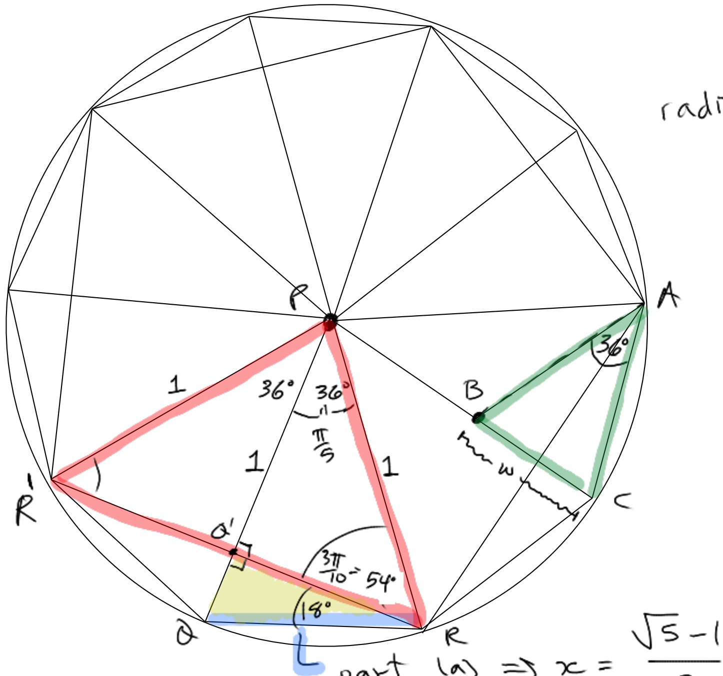
$\Rightarrow$ Isosceles $\Delta GHJ \Rightarrow \overline{GJ} = x$

Isosceles $\Delta GJF \Rightarrow \overline{FJ} = x$

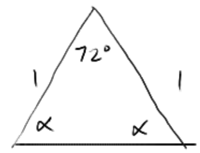
$\Rightarrow \overline{JH} = 1 - x$

Similar Δ's $\Rightarrow$

$$\frac{1}{x} = \frac{x}{1-x} \Rightarrow x = \frac{\sqrt{5}-1}{2}$$



radius = 1



$72 + 2\alpha = 180$

$\alpha = 54$

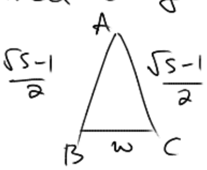
$72° - 54° = 18°$

part (a) $\Rightarrow x = \frac{\sqrt{5}-1}{2}$

① Start by in-laying unit-length isosceles 36° Δ @ center, 10 times to get decahedron.

② It's base length is $\frac{\sqrt{5}-1}{2}$ by (a)

③ Connect every other vertex for pentagon, making 18° angle. Double it to get ΔABC.



which is similar to first ΔPQR

④ Similar Δ's $\Rightarrow \dfrac{\frac{\sqrt{5}-1}{2}}{w} = \dfrac{1}{\frac{\sqrt{5}-1}{2}} \Rightarrow w = \left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{5-2\sqrt{5}+1}{4} = \frac{3-\sqrt{5}}{2}$

⑤ Finally, w is twice QQ', so $QQ' = \frac{3-\sqrt{5}}{4}$

⑥ So Pythag. $\Rightarrow (Q'R)^2 = \left(\frac{\sqrt{5}-1}{2}\right)^2 - \left(\frac{3-\sqrt{5}}{4}\right)^2 = \frac{5-2\sqrt{5}+1}{4} - \left(\frac{9-6\sqrt{5}+5}{16}\right)$

$$= \frac{24-8\sqrt{5}}{16} - \frac{14+6\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$$Q'R = \sqrt{\frac{10-2\sqrt{5}}{16}} = \sqrt{\frac{5-\sqrt{5}}{8}} = \frac{\sqrt{5-\sqrt{5}}}{\sqrt{8}} = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$$

⑦ and $Q'R$ is exactly ½ of side of inscribed Pentagon

$$\overline{RR'} = \sqrt{\frac{5-\sqrt{5}}{2}}$$

44. Explain how you could find 100 consecutive numbers, none of which is prime. How about a billion consecutive non-primes? (HINT: Factorials!)

(a)

$100!+1,\ 100!+2,\ 100!+3,\ \ldots\ 100!+100$

$100!+2 = 2\left(\frac{100!}{2}+1\right)$, $100!+3 = 3\left(\frac{100!}{3}+1\right)$, $100!+100 = 100(99!+1)$

or

$100!+100 = 100(99!+1)$

$100!+99 = 99(100\cdot 98!+1)$

$100!+98 = 98(100\cdot 99\cdot 97!+1)$

$\vdots$

(b) $10^9!+10^9$

$\vdots$

---

45. This is one of the most troubling unsolved problems in mathematics, for it is so ridiculously simple to state: any even number greater than or equal to 4 can be written as the sum of two primes. Alas, no one from Goldbach to Erdös has proved it.

45. (a) Verify the Goldbach Conjecture for 38, 538, and 1988.

(b) Is the conjecture true if we replace "even" with "odd"?

(c) Is the conjecture true if we replace "sum" with "product"?

$5=2+3$ ✓

$7=2+5$ ✓

$9=2+7$ ✓

$11=$ NO

$16=2^4$, not possible

46. Suppose that tomorrow someone proved the Goldbach Conjecture. Show that it would then follow that any integer (even or odd) greater than 5 can be written as the sum of three primes.

Assume Goldbach $\forall n\in\mathbb{Z}$, $\exists\ p,q$ prime s.t. $n=p+q$

Let $m>5$. If m is even $m=2+k$ w/ k is also even.

apply Goldbach to k: $m=2+p_1+p_2$ for some p_1, q_1 primes

If m is odd repeat for $m=3+k$ w/ k even.

49. Clearly, any positive integer must assume one of the six forms

$6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$, or $6k+5$

for some non-negative integer k.

(a) Use this to show that any prime greater than 3 is either one more or five more than a multiple of 6.

Let p = prime > 3.

$p \neq 6k, 6k+2, 6k+4$ (otherwise $6 \mid p$)

∴ $p = 6k+1$ or $p = 6k+5$

(b) Now suppose we say that if p, $p+2$, and $p+4$ are all prime, we'll call them "triplet primes." While it is unknown whether there are finitely or infinitely many pairs of prime twins, we can determine whether the following conjecture is true or false:

"There are infinitely many prime triplets."

Do so, using (a).

If $p, p+2, p+4$ are all prime. then $p \neq 2$.

there are two cases:

1 $p = 6k+1$, 2 $p = 6k+5$

case 1 $p+2 = 6k+1+2 = 6k+3 = 3(k+1)$ (not prime)

case 2 $p+4 = 6k+5+4 = 6k+9 = 3(2k+3)$ not prime

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(a) $(4m+1)(4m+1) = 16m^2 + 8m + 1 = 4M+1$

(b) Let $\{a,b,c,\dots,e\}$ be all the $4m+3$ primes. (finite)

Let $f = 4(abc\cdot\dots\cdot e) - 1$. Since $f = 4M+3$ f is either prime or some prime divides it.

$(4m+3)(4m+3) = 16m^2 + 24m + 9 = 4M + 8 + 1 = 4K+1$

$f = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_n^{k_n} = 4(abcd\cdot\dots\cdot e) - 1$

if all the p_i are $4m+1$ primes then the LHS is of form $4m+1$ by (a). So some prime $p = 4m+3$ divides f.

but p is on the list $a,b,\dots,e$ so it also divides

$l = abc\cdot\dots\cdot e$

$\Rightarrow p|a, p|b \Rightarrow a = pq_1$ + $b = pq_2 \Rightarrow b-a = p(q_2-q_1) \Rightarrow p | b-a$

$\Rightarrow$ p divides $l - f = 1$. ⊗

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$p = 4n+3$

$p = a^2 + b^2$

If $a = 2n$, $b = 2m$

$\Rightarrow 4n+3 = 4n^2 + 4m^2 = 4(n^2+m^2)$ ⊗

If $a = 2n+1$, $b = 2m$

$4n+3 = 4n^2 + 4n + 1 + 4m^2 = 4k+1$ ⊗

If $a = 2k+1$, $b = 2n+1$

$4n+3 = 4k^2 + 4k + 1 + 4n^2 + 4n + 1 = 4M + 2$ ⊗