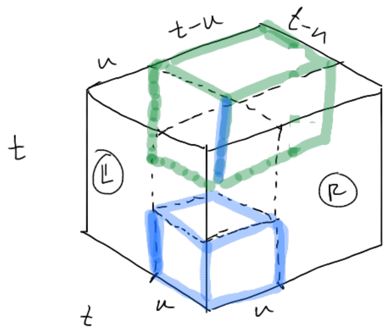
Cardono's sol'n of the depressed cubic



geometry $\dashrightarrow$

$$t^3 = u^3 + (t-u)^3 + \overset{L+R}{2tu(t-u)} + \overset{\text{above blue}}{u^2(t-u)} + \underset{\text{below green}}{u(t-u)^2}$$

algebra $\downarrow$

$$t^3 - u^3 = (t-u)^3 + (t-u)[\underbrace{2tu + u^2 + u(t-u)}_{3tu}]$$

$$\boxed{t^3 - u^3 = (t-u)^3 + (t-u)(3tu)} \leftarrow \text{geometry + algebra}$$

$\updownarrow$

write

$x = t-u$

$3tu = m \quad (1)$

$n = t^3 - u^3 \quad (2)$

$$\boxed{n = x^3 + x \cdot m}$$

depressed cubic

$$(1)\ u = \frac{m}{3t} \overset{\text{sub}}{\rightsquigarrow} (2)\quad n = t^3 - \left(\frac{m}{3t}\right)^3 = t^3 - \frac{m^3}{27t^3} \Rightarrow n = t^3 - \frac{m^3}{27t^3}$$

[now mult. by t^3]

$$\boxed{t^6 - nt^3 - \frac{m^3}{27} = 0}$$

quadratic in t^3

$\frac{1}{2} = \frac{1}{\sqrt{4}}$

$$t^3 = \frac{n + \sqrt{n^2 + \frac{4}{27}m^3}}{2} = \frac{n}{2} + \frac{1}{2}\sqrt{n^2 + \frac{4}{27}m^3}$$

$$t^3 = \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}, \text{ so } \boxed{t = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}}$$

by (2) $\Rightarrow u^3 = t^3 - n$ so

$$= \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}} - n \Longrightarrow$$

$$\boxed{u = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}} \Rightarrow \boxed{x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}}$$

Recall:

$w^4 - 7w^2 + 12 = 0$

$(w^2)^2 - 7(w^2) + 12 = 0$

quadratic in w^2 ___

q.F. $\Rightarrow w^2 = \frac{7 \pm \sqrt{49-48}}{2} = \frac{7 \pm 1}{2} = \frac{8}{2} = 4$ or $\frac{6}{2} = 3$

$w^2 = 4, w = \pm 2$

$w^2 = 3 \quad w = \pm\sqrt{3}$

Ex. Solve $9x^3 - 9x = 4 \rightsquigarrow x^3 - x = \frac{4}{9}$

Form: $x^3 + mx = n$

$\Rightarrow \boxed{m = -1,\ n = \frac{4}{9}}$

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

$$= \sqrt[3]{\frac{2}{9} + \sqrt{\frac{4}{81} - \frac{1}{27}}} - \sqrt[3]{-\frac{2}{9} + \sqrt{\frac{4}{81} - \frac{1}{27}}}$$

$$= \sqrt[3]{\frac{2}{9} + \sqrt{\frac{4}{81} - \frac{1}{27}\cdot\frac{3}{3}}} - \sqrt[3]{-\frac{2}{9} + \sqrt{\frac{4}{81} - \frac{1}{27}\cdot\frac{3}{3}}} = \sqrt[3]{\frac{2}{9} + \underbrace{\sqrt{\frac{1}{81}}}_{\frac{1}{9}}} - \sqrt[3]{-\frac{2}{9} + \underbrace{\sqrt{\frac{1}{81}}}_{\frac{1}{9}}}$$

$$= \sqrt[3]{\frac{3}{9}} - \sqrt[3]{-\frac{1}{9}} = \frac{\sqrt[3]{3} - \sqrt[3]{-1}}{\sqrt[3]{9}} \cdot \frac{9^{2/3}}{9^{2/3}} = \frac{3^{1/3}\cdot(3^2)^{2/3} + 3^{4/3}}{9} = \frac{3^{5/3} + 3^{2/3}}{3^2} = \frac{3(3^{2/3} + 3^{1/3})}{3^2}$$

$$= (3^{2/3} + 3^{1/3})\,3^{-1} = \underbrace{3^{-1/3} + 3^{-2/3}}_{\text{Cardano}} = u + u^2$$

$$= 3^{-1/3} + 3^{-2/3}$$

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check: set $u = 3^{-1/3}$, so $u^3 = 3^{-1} = \frac{1}{3}$, $u^4 = 3^{-4/3}$, $u^5 = 3^{-5/3}$, $u^6 = 3^{-2}$

$x^3 - x = \frac{4}{9}$ idea: $\boxed{x = u + u^2}$

1
1 1
1 2 1
1 3 3 1
degree sum constant

$$(u + u^2)^3 - (u + u^2) \stackrel{?}{=} \frac{4}{9}$$

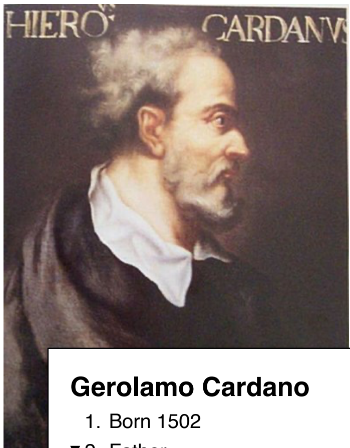
$$\downarrow$$

$$u^3 + 3\underbrace{u^2 u^2}_{u^4} + 3\underbrace{u(u^2)^2}_{u^5} + (u^2)^3 - u - u^2$$

$$\frac{1}{3} + 3\cdot 3^{-4/3} + \underbrace{3\cdot 3^{-5/3}}_{+3^{-2/3}} + 3^{-2} - 3^{-1/3} - 3^{-2/3}$$

$$\frac{1}{3} + 3^{-2} = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$$

☺



Gerolamo Cardano

1. Born 1502
2. ▼ Father
    a. Fazio Cardano
    b. Student of perspective
    c. ▼ Professor
        i. ▼ hermetical science
            1. Based on purported teachings of Hermes
            2. primeval divine wisdom, revealed to the few
    d. ▼ Devoted to the world of the occult.
        i. Supernatural beliefs outside of science & religion
        ii. had a “familiar” spirit
    e. Friend of Leonardo Da Vinci
3. ▼ Mother
    a. Tried to abort Gerolamo