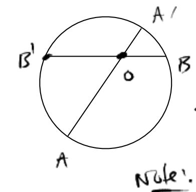
Hyperbolic Geom & Inversion in Circles.

Reminder: Cross Ratio $(AB, PQ) = \dfrac{\overline{AP}\,\overline{BQ}}{\overline{BP}\,\overline{AQ}}$.

Power of a point O wrt circle:

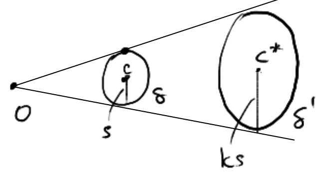


$\overline{OA'}\cdot\overline{OA} = r^2 = \overline{OB'}\cdot\overline{OB}$

Note: O can be inside or outside circle.

Dilation: w/ center O, ratio k is the transformation in Eucl. plane where

0



$C \longmapsto C^* = k\cdot C$

$O \longmapsto O$

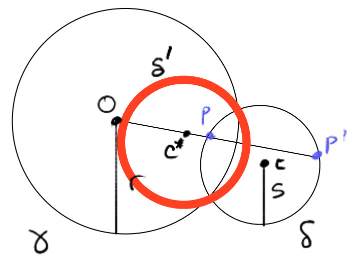
points are moved radially outward from O.

Prop. 7.6: Inversion in γ. Let δ be another circle.

1. Set $p =$ power of O wrt δ.

then $k = \dfrac{r^2}{p}$ inversion radius, $p = \overline{OP}\cdot\overline{OP'}$

wrt δ, other circle



2. If $\delta' =$ image of δ under inversion in γ.
   - It's radius is $k\cdot s$
   - It's center C^* image of C under dilation from O w/ ratio k.

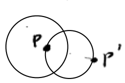
Cor: Circle $\delta \perp \gamma \iff \delta$ is fixed under inversion in γ.

proof:

$(\Longrightarrow)$ If $\delta \perp \gamma$ then $\overline{OP}\cdot\overline{OP'} = p = r^2$

(Prop. 7.5 $\Rightarrow$) $P' \in \delta$ $p =$ power of O wrt γ, $r =$ radius of γ, P' inverse of P.

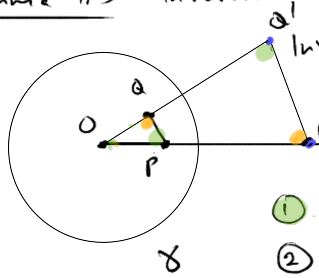
$\Rightarrow k = \dfrac{r^2}{p} = \dfrac{r^2}{r^2} = 1 \Rightarrow \delta = \delta'$.



$(\Longleftarrow)$ If δ fixed by inversion in γ.

$\delta = \delta' \Rightarrow p = r^2$ & δ contains P' (7.5) $\Rightarrow \delta \perp \gamma$.

Lemma 7.3: Inversion & Similarity (Congruent angles in Δ's)



Invert about γ. If P, Q & not collinear w/ O then let P', Q' be inverse of P, Q resp.

$\Delta OPQ \sim \Delta OQ'P'$

① $\angle O$ common to both

② $\overline{OQ} \cdot \overline{OQ'} = r^2 = \overline{OP} \cdot \overline{OP'}$ (power invariance)

③ divide: $\dfrac{\overline{OQ}}{\overline{OP'}} = \dfrac{\overline{OP}}{\overline{OQ'}}$ (① + ③ Similar Δ condition)

$\Rightarrow$ The two triangles are similar

Prop. 7.7: Lines outside γ, under inversion in γ, go to: Punctured circles inside γ & the puncture is @ C.

Note: C has no image in plane. (think: $C \mapsto \infty$)