Def'n: <u>Projective Plane</u> (replace E. // property w/ Elliptic)

model of <u>incidence</u> <u>geometry</u> with
1. no parallel lines
2. at least 3 points on every line.

point at infinity: point where // lines meet.

Idea:

<u>Railroad Tracks</u>:

In reality:

They look like

we add point where they intersect

<u>Note</u>: This extension of the Euclidean Plane uses only incidence axioms
- not betweenness or congruence & is called <u>AFFINE GEOMETRY</u>.

BTW — what is "geometry"?
- relations b/w points, lines, planes, ...

<u>Ex</u>.

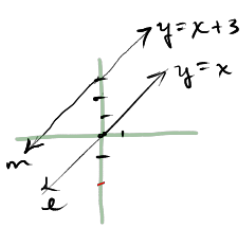
Affine Plane

<u>Note</u>: To extend A.P to Proj. Plane we add enough points at infinity to make parallel lines intersect.
These points at infinity need to be connected by a line. (Axiom I.)

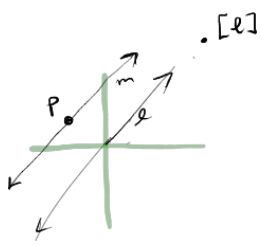
What are points & lines? "We are free to interpret lines & points as we wish as long as we can prove the axioms are satisfied in that interpretation". (p. 82)

<u>Ex</u>. Extending Real Affine Plane (regular cartesian plane without distance/angle machinery)

$l \sim m$ if they're parallel, so they lie in same equivalence class, say, $[l]$
↳ equivalence relation



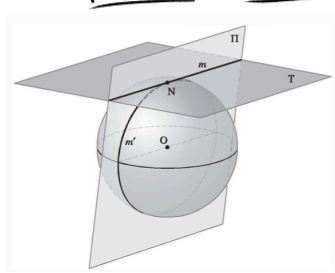
Point at Infinity $\equiv [l]$
↳ we define new "line" to include it's equivalence class as a point.
[actually $[l]$ is a point ... at infinity]



Although we can't draw it — l & m meet at $[l]$.
- l & m meet $[l]$ from two different directions.

l_∞ = line at infinity $\equiv$ set of all points at ∞.

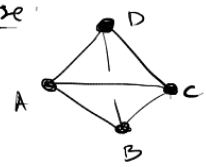
this process is called the <u>projective</u> <u>completion</u>

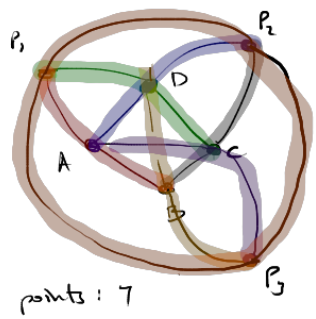


<u>Q</u>: what line does P & $[l]$ determine?

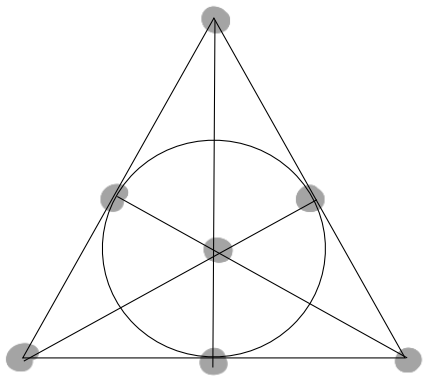
<u>Q</u>: why does every line contain 3 points?

Exercise: Projectively complete this affine plane.





points : 7
lines : 7



smallest projective plane

Fano Plane

Algebraic Models of Affine Planes

F = field : $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_2$ ← $\{0,1\}$ w/ $+$ & $\cdot$ mod 2.

F^2 = ordered pairs of element of F. (algebra)

↳ can be viewed as affine plane by:

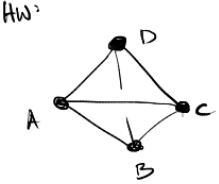
points = pairs

lines = solutions to $ax + by + c = 0$ (a, b, c not all zero)

Incidence: point lies on line if it solves the eq'n.

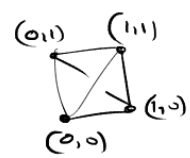
(Affine Plane over the field F)

Ex. HW:



Field = $\mathbb{Z}_2$

$F^2 = \{(0,0), (0,1), (1,0), (1,1)\}$



$ax + by + c = 0$

$(0,1)$, $x=0$, $y=1$ | $a \cdot 0 + b \cdot 1 + c = 0$

$b + c = 0$ work in $\mathbb{Z}_2$.

$b = -c$ ← additive inverse of c. ($1 + 1 = 2 \bmod 2 = 0$)

$\left.\begin{matrix} a=0 \\ b=1 \\ c=1 \end{matrix}\right\}$ $y + 1 = 0$ ($y = -1$ but $1 = -1$ here!)

$\Rightarrow$ $(0,1)$ lies on $\overset{[a,b,c]}{[0,1,1]}$

$(1,1)$, $x=1$, $y=1$: $0 \cdot 1 + 1 \cdot 1 + 1 = 2 \bmod 2 = 0$ $\Rightarrow$ $(1,1)$ lies on $[0,1,1]$ too

so $(0,1)$ & $(1,1)$ determine $[0,1,1]$