Wed. Wk 13

$SL_2(\mathbb{R})$ = 2 × 2 matrices, real entries

Special Linear – matrix group

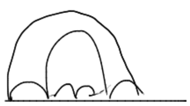
Special ⟷ determinant = 1

$PSL_2(\mathbb{R}) = SL_2\mathbb{R} / \pm I$

projective

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $ad - bc = 1$ ⟹ 3-manifold

IS isomorphic to the group of Hyperbolic, orientation preserving isometries



<u>Almost</u> the group of hyp. isometries

$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \in SL_2(\mathbb{Z}) \subset SL_2\mathbb{R}$

⟷ mobius transf.

$\dfrac{z+2}{2z+5}$ ⟵ hyperbolic isometry

but this association isn't unique since

$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix} \xrightarrow{\det = 1}$ so this is also in $SL_2\mathbb{R}$

⟷ and gives the same Mobius Tran

$$\frac{-z-2}{-2z-5} = \frac{z+2}{2z+5}$$

Thm: $SL_2\mathbb{R}$ is generated by two elements, a translation and φ = inversion about unit circle followed by a reflection through $x=0$

Proof

$$T_a(x,y) = (x+a, y) \leftrightarrow \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = T_a \qquad \mathbb{R}^2 \cong \mathbb{C} \quad (\text{set } z = x+iy)$$

$$\varphi(x,y) = \left(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \sim \frac{-x+iy}{x^2+y^2} = -\frac{\cancel{(x-iy)}}{(x+iy)\cancel{(x-iy)}} = \frac{-1}{z}$$

$$= \frac{0z-1}{1z+0} \Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma$$

Look @ three compositions:

$$\sigma \cdot T_r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & r \end{pmatrix}$$

$$\sigma \cdot T_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix}$$

$$\sigma \cdot T_t \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}$$

So we must show any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$ can be written like this $\leftarrow$

Finally,

$$(\sigma \cdot T_t)(\sigma \cdot T_s)(\sigma \cdot T_r) = \begin{pmatrix} -s & 1-rs \\ st-1 & rst-r-t \end{pmatrix}$$

So express r, s, t via a, b, c.

For $a \neq 0$, set $\boxed{-s = a}$, so $1 - rs = b \Rightarrow 1 + ra = b$ thus $\boxed{r = \frac{b-1}{a}}$

continue to get t via a, b, c, d.

Cor

Since translations & circle inversions map {circles, lines} to {circles, lines} $SL_2\mathbb{R}$ preserves circles & lines

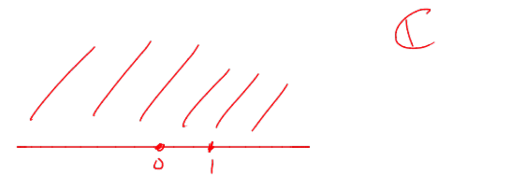
Cross-Ratio

Let $a, b, c, d \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (@ least 3 distinct)

$$[a,b;c,d] = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

$$[a,b;c,a] = \frac{(a-c)(b-a)}{(a-a)(b-c)} = \infty$$

thm Hyp. Isom preserve the cross-ratio.



Commonly, think of 1st coord as variable

$$[z, \underset{\downarrow \text{ to } 1}{a}; \underset{\downarrow \text{ to } 0}{b}, \overset{\nearrow \infty}{c}] = \frac{(z-b)(a-c)}{(z-c)(a-b)} = \text{Mobius Transf}$$

Note: sub $z = a \Rightarrow$ get output $= 1$

$z = b \rightarrow$ get 0

Ex Create a Mobius Trans sending $1 \longmapsto 1$, $-i \longmapsto 0$, $-1 \longmapsto \infty$

$$[z, 1, -i, -1] = \frac{(z+i)(1+1)}{(z+1)(1+i)} = \frac{2z+2i}{(1+i)z + (1+i)} = f(z)$$

verify:

$$f(-1) = \frac{-2+2i}{0} = \infty \quad \Big| \quad f(1) = \frac{2+2i}{(1+i)+(1+i)} = 1 \quad ☺$$

Ex. $i \mapsto 1,\ \infty \mapsto 2,\ 0 \mapsto 3$

want

$\gamma = \gamma_2^{-1} \circ \gamma_1$

γ_1

$i \mapsto 1$

$\infty \mapsto 0$

$0 \mapsto \infty$

γ_2

$1 \mapsto 1$

$2 \mapsto 0$

$3 \mapsto \infty$

$[z, i, \infty, 0] = \gamma_1(z)$, with $i \to 1$, $\infty \to 0$, $0 \to \infty$

$[w, 1, 2, 3] = \gamma_2(w)$

check

$\gamma(i) = \gamma_2^{-1} \cdot \gamma_1(i) = \gamma_2^{-1}(1) = 1$

$\gamma(\infty) = \gamma_2^{-1} \gamma_1(\infty) = \gamma_2^{-1}(0) = 2$

$\gamma(0) = 3$

How to find the mobius tra?

set $\gamma(z) = w$

$\gamma_2^{-1} \cdot \gamma_1(z) = w$

$\Rightarrow \gamma_1(z) = \gamma_2(w)$

$$\boxed{[z, i; \infty, 0] = [w, 1; 2, 3]}$$

goal: solve for w in terms of z

$$\frac{(z-\infty)(i-0)}{(z-0)(i-\infty)} = \frac{(w-2)(1-3)}{(w-3)(1-2)}$$

$$\frac{i}{z} = \frac{4-2w}{3-w} \Rightarrow 3i - wi = 4z - 2wz$$

$$3i - 4z = wi - 2wz = w(i - 2z)$$

$$\frac{3i-4z}{i-2z} = w = \gamma(z)$$

$$\gamma(i) = \frac{3i - 4i}{i - 2i} = \frac{-i}{-i} = 1$$

$SL_2\mathbb{R}$ acts transitively on $\mathbb{H}^2$
group space

group G acts on space X means $\forall g \in G$ and $x \in X$ $\exists$ $x_g \in X$

$x \xrightarrow{g} x_g$

transively:

$G \circlearrowright X$ transitively if $x, y \in X$ $\exists$ $g \in G$ s.t. $g(x) = y$.

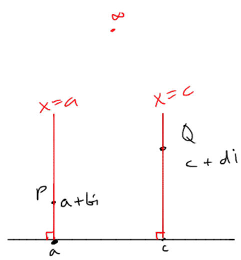
you can from any point in X to any other by an element in G

Theorem: Given any two points $P, Q \in \mathbb{H}^2$ $\exists$ isometry st $f(P) = Q$.

$P = a+bi$, $Q = c+di$.

Choose one that fixes ∞, i.e., $f(\infty)=\infty$ & $f(a+bi) = c+di$
so f takes the $x=a$ line to $x=c$ line (f preserve angles)

i.e., $f(a) = c$



$$[z, a+bi; a, \infty] = [w, c+di; c, \infty]$$

$$\frac{z-a}{bi} = \frac{w-c}{di} \quad \text{so} \quad (w-c)bi = di(z-a)$$

$$(w-c)b = d(z-a)$$

$$\boxed{f(z) = w = \frac{d(z-a)}{b} + c}$$

$$= \frac{dz - da}{b} + \frac{cb}{b}$$

$$= \frac{d}{b}z + \frac{cb-da}{b}$$

$$\updownarrow \begin{pmatrix} d/b & cb-da/b \\ 0 & 1 \end{pmatrix}$$

Note: $d>0, b>0$ so $\det > 0$
so this gives a hyp isom

same map after scaling $\begin{pmatrix} d & cb-da \\ 0 & b \end{pmatrix}$

What are fixed points of our map?

$$z = \frac{d(z-a)}{b} + c \implies (z-c)b = d(z-a)$$

$$bz - bc = dz - da$$

$$da - bc = z(d-b)$$

If $b = d \Rightarrow z = \infty$ which corresponds to a translation w/ fixed pt @ ∞

$$\boxed{\frac{da-bc}{d-b} = z}$$

If $b \neq d$ then $\frac{da-bc}{d-b} \in \mathbb{R}$ since a, b, c, d are all real,

$\Rightarrow$ No fixed points in $\mathbb{H}^2$

$\mathbb{H}^2$

fixed axis

Ex.

Model.

$$\begin{pmatrix} d & cb-da \\ 0 & b \end{pmatrix}$$

Question: give a Mobius transformation taking

$$\underset{a}{5} + \underset{b}{7}i \text{ to } \underset{c}{3} + \underset{d}{6}i$$

$$\updownarrow$$

$$\begin{pmatrix} 6 & 3\cdot 7 - 6\cdot 5 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 6 & -9 \\ 0 & 7 \end{pmatrix}$$

$$\updownarrow$$

$$f(z) = \frac{6z-9}{7} = \frac{6}{7}z - \frac{9}{7}$$

$$f(5+7i) = \frac{6(5+7i)}{7} - \frac{9}{7} = \frac{30+42i-9}{7} = \frac{21+42i}{7} = 3+6i$$

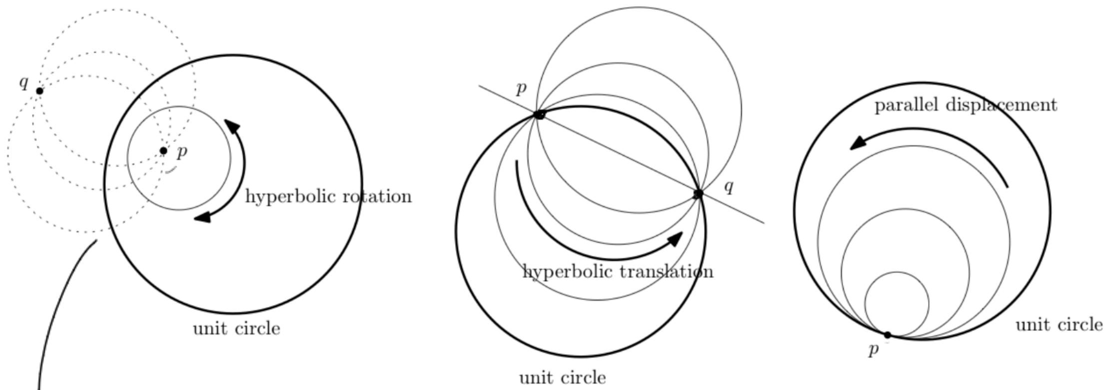


Figure 3.3.7. Three types of hyperbolic transformations