Mon Wk 14

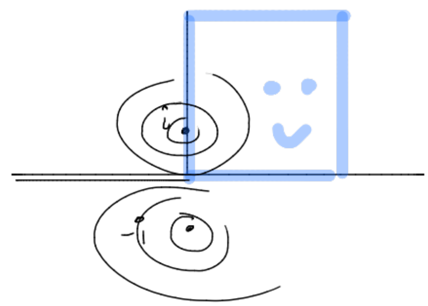
Finish Royster

P. 122

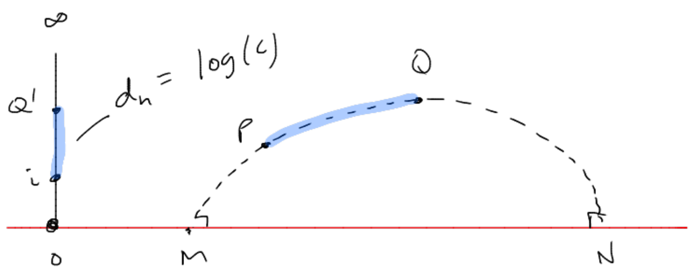
Ellipic

$$\varphi(z) \longleftrightarrow \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \text{ by } \varphi(z) = \frac{\cos(\beta)\cdot z - \sin(\beta)}{\sin\beta\, z + \cos\beta}$$

$$\varphi(i) = \frac{\cos(\beta)\cdot i - \sin\beta}{\sin(\beta)\, i + \cos\beta} \cdot \frac{\sin(\beta)\, i - \cos\beta}{\sin(\beta)\, i - \cos\beta} = \frac{-\cos\beta\sin\beta - (\cos^2\beta + \sin^2\beta)\, i + \sin\beta\cos\beta}{-\sin^2\beta - \cos^2\beta} = i$$



How to find H-dist b/w points P, Q



How to define Mobius Trans that sends

$M \mapsto 0$, $P \mapsto i$, $Q \to Q'$, $N \to \infty$

(3 suffice)

$(z, P; M, N) = (w, i; 0, \infty)$ ($w \in$ on imaginary axis) $w = ci$

$= (ci, i; 0, \infty) = \dfrac{(ci - 0)\cancel{(i-\infty)}}{\cancel{(ci-\infty)}(i-0)} =$

so $z = Q \Rightarrow$

$(Q, P; M, N) = c.$

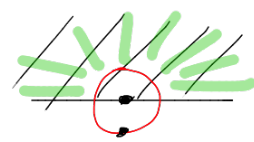
Interpret c

$\log(c)$ is the H-dist b/w i and $Q' = ci$.

So $d_h(P, Q) = \log(c) = (Q, P; M, N)$

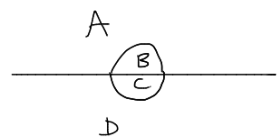
Connect $\mathbb{H}$ w/ Poincaré disk

Use mobius transformation that sends $\mathbb{H} \longleftrightarrow$ Unit circle



$$\varphi = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

$$\varphi(z) = \frac{z - i}{1 - iz}$$



$A \mapsto B$
$B \mapsto C$
$C \mapsto D$
$D \mapsto A$

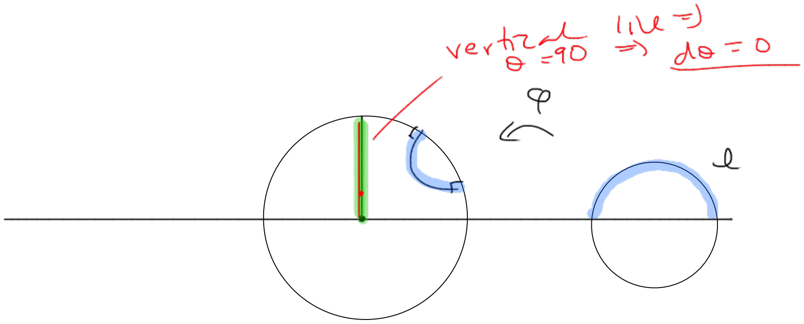
$$\varphi(0) = \frac{0 - i}{1} = -i$$

$$\varphi(i) = \frac{i - i}{1 - i^2} = \frac{0}{2} = 0$$

$$\varphi(-i) = \frac{z - (-i)}{1 - i(-i)} = \frac{z + i}{0} = \infty$$

$$\varphi(\infty) = \frac{\infty - i}{1 - i\infty} = \frac{\infty}{-i\infty} = \frac{1}{-i} = i$$

To see where ℓ goes consider $\varphi(\mathbb{R})$

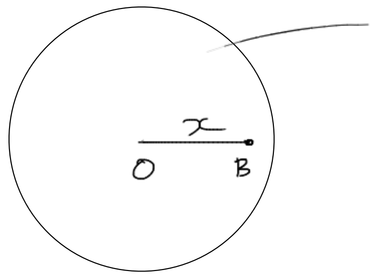


Usually coords in Poincaré disk: $P = \{re^{i\theta} \mid 0 \leq r < 1\}$

$$ds = \frac{2\sqrt{dr^2 + r^2 d\theta^2}}{1 - r^2} \quad \text{arc length}$$

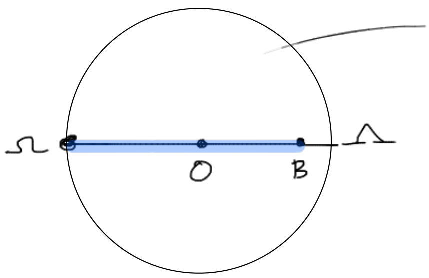
Lemma

Let O = origin, If $d_p(O, B) = x$ then



$$d(O,B) = \frac{1-e^x}{e^x+1}$$

proof



let Ω, Λ be ends of diameter thru OB

$$x = \log(O, B; \Omega, \Lambda)$$

$$e^x = (O, B; \Omega, \Lambda)$$

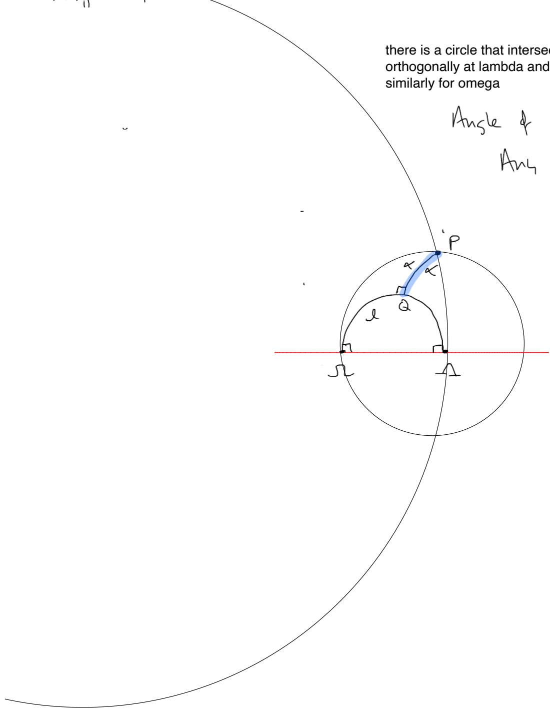
$$= \frac{(O-\Omega)(B-\Lambda)}{(O-\Lambda)(B-\Omega)} = \frac{O\Omega \; B\Lambda}{O\Lambda \; B\Omega} = \frac{B\Lambda}{B\Omega}$$

$$= \frac{1-OB}{1+OB}$$ solve for OB

$$e^x + e^x OB = 1 - OB \implies (e^x+1)OB = 1 - e^x$$

$$OB = \frac{1-e^x}{1+e^x}$$

Angle of Parallelism



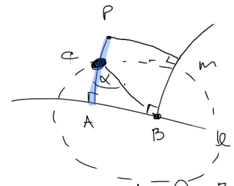
there is a circle that intersects the boundary orthogonally at lambda and also goes thru P, and similarly for omega

Angle of Parallelism only depends on $d(P,Q)$.

Any line thru P making angle w/ PQ less than α intersects ℓ &, greater than α doesn't.

Bolyai's construction was

1. let P not lie on ℓ
2. drop ⊥ from P to ℓ, making pt A.
3. choose $B \in ℓ$ raise ⊥ from ℓ called m
4. drop ⊥ from P to m
5. PM is longer than AB (hyperbolic)



6. Form circle @ B of radius PA
7. intersectn of circl w/ PA gives C giving angle of parallelism