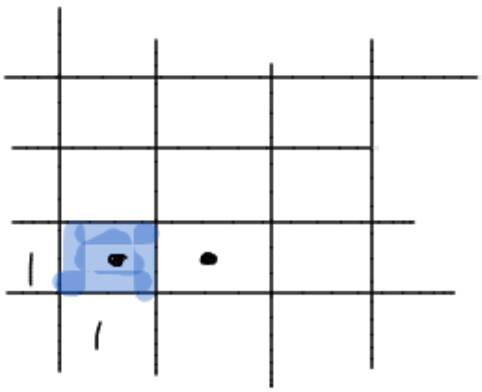


$f(\bar{x}) = \bar{x} + \begin{pmatrix}0\\1\end{pmatrix}$ $\qquad \uparrow \qquad$ $\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$

$g(\bar{x}) = \bar{x} + \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad \longrightarrow \qquad$ $\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}$

$\langle f, g \rangle$ = Subgroup

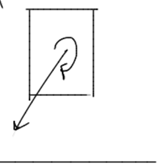
Normal Form of orthogonal Matrices

$$\begin{bmatrix} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & -1 & 0 & & & \\ & & 0 & -1 & & & \\ & & & & \boxed{\begin{matrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{matrix}} & & \\ & & & & & \begin{matrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{matrix} & \\ & & & & & & \ddots \end{bmatrix}$$

3 dim's that act as axes of rotations

reflections

Rot



See p.10 to know where

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ (rotation) comes from. note determinant.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ (reflection)

Frames p.14

Euclidean Frame: $\mathbb{R}^n$

$n+1$ points, $Q_0, Q_1, \ldots, Q_n$, $\overrightarrow{Q_0Q_i} \perp \overrightarrow{Q_0Q_j}$ w/ $d(Q_0, Q_i) = 1$

thm: If we fix a frame

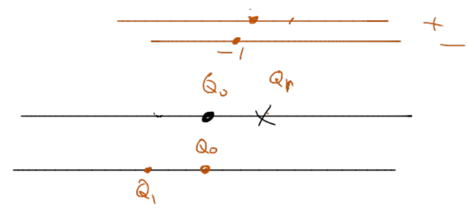
the set of motions (isometries) $\xleftrightarrow{1-1}$ Euclidean Frames

Ex. $n=1$,

{Set of Motions of $\mathbb{R}$} $\longleftrightarrow$ Euclidean 1-frames

How many?

$Q_0 \in \mathbb{R}$



$x \longmapsto x+1$

$x \longmapsto -x+1$

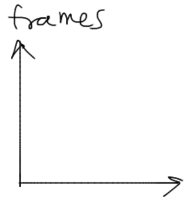
Ex $n=2$ $\mathbb{R}^2$

{Set of Motions of $\mathbb{R}^2$} $\longleftrightarrow$ Euclidean 2-frames

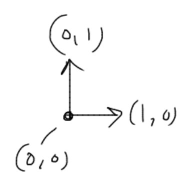
$\left\{ \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}, \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \right\}$

$\begin{pmatrix} p_i \\ q_i \end{pmatrix} \cdot \begin{pmatrix} p_j \\ q_j \end{pmatrix} = 0$

$i \neq j$



Frame #1



Any rotation & fix (0,0), still get frame

$S^1 \times \mathbb{R}^2 \cup S^1 \times \mathbb{R}^2$

circle

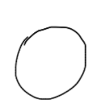
S^n = set of pts in $\mathbb{R}^{n+1}$, norm = 1

$$\boxed{\begin{matrix} \text{Ex } (\theta, x_1, y_1) \in S^1 \times \mathbb{R}^2 \\ \Updownarrow \\ \text{rot}_0(\theta) \text{ \& translate by } (x,y) \end{matrix}}$$

Ex $n=3$ $\mathbb{R}^3$

{Set of Motions of $\mathbb{R}^3$} $\longleftrightarrow$ Eucl. 3-frames $\longleftrightarrow$ $S^2 \times \mathbb{R}^3 \cup S^2 \times \mathbb{R}^3$

$$T(x) = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{x} \end{pmatrix} + \bar{b}$$

 $\longleftrightarrow$ set of 2-D rotations

Motions are rigid,

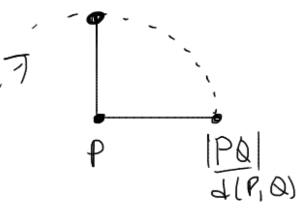
Prop: A motion of $\mathbb{E}^2$ is determined by what it does to 3 points, P,Q,R & what it does to 2 points leaves only 2 possibilities.

Ex Let's prove the 2nd Half first.

Suppose $P \longrightarrow P'$, $Q \longrightarrow Q'$,

Fix frame P_0, P_1, P_2 s.t.

$P_0 = P$, $P_1 = \dfrac{|PQ|}{d(P,Q)}$, $P_2 =$ something that works



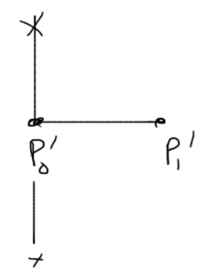
Note that I have now fixed a frame ——— (see previous Prop.)

Now fix another frame

$P_0' = P'$

$P_1' = \dfrac{|P'Q'|}{d(P',Q')}$, How many choices do I have?

$P_2' =$ 2 choices



Prop Any motion of $\mathbb{E}^2$ is either

- $\mathrm{rot}_p(\theta)$
- $\mathrm{refl}(L)$
- $\mathrm{trans}(\bar{b})$
- $\mathrm{glide}(\bar{v})$

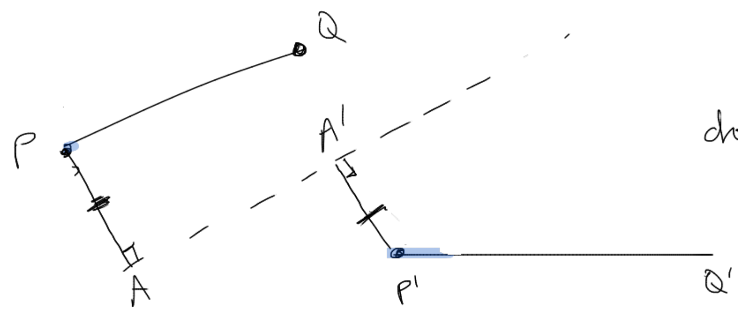
proof: Let $P \longmapsto P'$, $Q \longmapsto Q'$ via some motion, 2 possibilities.

$\bar{u} = PQ$

$\bar{v} = P'Q'$

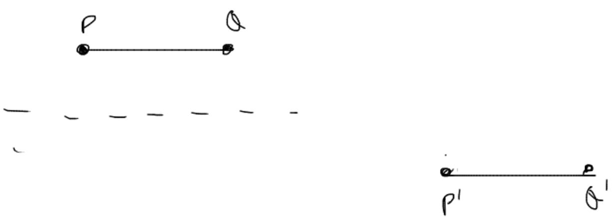
θ = angle b/w P, Q

$\frac{1}{2}(\bar{u}+\bar{v})$ = vector spans line "L" to reflect, taking $PQ \parallel P'Q'$



drop $\perp$ from P, P' to L determines A, A'

trans vector $\overline{AA'}$ takes $P \to P'$



For a rotation taking