HW Review

$\cos\theta = \dfrac{a \cdot b}{|a| \cdot |b|}$ w/ 

use Schwartz Inequality

1.5/ Let $\lambda = \cos\theta + i\sin\theta$, $\theta \neq 0, \pi$, $A\bar{z} = \lambda\bar{z}$, $\bar{z} = \bar{x} + i\bar{y}$. Show real & imag parts are equal in length & orthogonal $|x|^2 = |y|^2$, $x \cdot y = 0$

$A\bar{z} = A(x+iy) = A\bar{x} + iA\bar{y}$

$= \lambda\bar{z} = (\cos\theta + i\sin\theta)(\bar{x} + i\bar{y})$

$= \cos\theta\bar{x} + i\cos\theta\bar{y} + i\sin\theta\bar{x} - \sin\theta\bar{y}$

$= \underbrace{(\overbrace{\cos\theta\bar{x} - \sin\theta\bar{y}}^{\text{vector}})}_{A\bar{x}} + \underbrace{(\sin\theta\bar{x} + \cos\theta\bar{y})}_{A\bar{y}}i$

$A \in O(n)$, $|A\bar{x}|^2 = |\bar{x}|^2$,

$A\bar{x} \cdot A\bar{x} = (\cos\theta\bar{x} - \sin\theta\bar{y}) \cdot (\cos\theta\bar{x} - \sin\theta\bar{y})$

$|x|^2 = \cos^2\theta|\bar{x}|^2 - 2\sin\theta\cos\theta\bar{x}\cdot\bar{y} + \sin^2\theta|\bar{y}|^2$

$(1-\cos^2\theta)|x|^2 = -2\sin\theta\cos\theta\bar{x}\cdot\bar{y} + \sin^2\theta|y|^2$

$\Rightarrow \bar{x}\cdot\bar{y} = 0$

$\boxed{\begin{gathered}\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}4\\5\end{pmatrix} = 23 \\ \left(\begin{pmatrix}5\\5\end{pmatrix}-\begin{pmatrix}3\\2\end{pmatrix}\right)\left(\begin{pmatrix}6\\6\end{pmatrix}-\begin{pmatrix}2\\1\end{pmatrix}\right) \\ = \begin{pmatrix}5\\5\end{pmatrix}\cdot\begin{pmatrix}6\\6\end{pmatrix} - \begin{pmatrix}5\\5\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}3\\2\end{pmatrix}\begin{pmatrix}6\\6\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} \\ 60 - 15 - 30 + 8 \\ 30+8-15 = 23\end{gathered}}$

Now recall that $A \in O(n)$ preserves angles too, i.e., dot product

i.e., Motions $\longleftrightarrow$ orthogonal Matrices $\longrightarrow$ preserve dot

= rigid distance $\leftrightarrow$ dot product

& angle preserving transformation

dot product

(1) $A\bar{x} \cdot A\bar{y} = \bar{x}\cdot\bar{y}$ b/c $A \in O(n)$

(2) $\bar{x}\cdot\bar{y} = 0$

$\bar{x}\cdot\bar{y} = A\bar{x}\cdot A\bar{y} = \cos\theta\sin\theta|x|^2 + (\cos^2\theta - \sin^2\theta)\bar{x}\cdot\bar{y} - \sin\theta\cos\theta|\bar{y}|^2$

$0 = \sin\theta\cos\theta(|x|^2 - |\bar{y}|^2)$

$\Rightarrow |\bar{x}|^2 = |\bar{y}|^2$

1.6/

(a) Let A be a real orthogonal matrix and let e,f be real eigenvectos of A with distinct eigenvalues, lambda and mu. Show that e and f are orthogonal.

(b) $z \in \mathbb{C}^n$ eigenvector, w/ complex eigenvalue $\lambda \notin \mathbb{R}$

show $\bar{z} \cdot \bar{z} = 0$

Fact $|\lambda| = 1$ (step 2 p.12) $\theta \neq 0, \pi$

$\theta = 0$

$\Rightarrow$

$\lambda = \cos\theta + i\sin\theta$

$\bar{z} \cdot \bar{z} = A\bar{z} \cdot A\bar{z} = \lambda\bar{z} \cdot \lambda\bar{z} = \lambda^2 \bar{z} \cdot \bar{z}$ ★

$\lambda^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$= \cos^2\theta - \sin^2\theta + \underbrace{(2\sin\theta\cos\theta)}_{\neq 0} i$

$\bar{z} \cdot \bar{z} - \lambda^2 \bar{z} \cdot \bar{z} = 0$

$\underbrace{(1-\lambda^2)}_{\neq 0}(\bar{z} \cdot \bar{z}) = 0 \Rightarrow \bar{z} \cdot \bar{z} = 0$

More on $\bar{z} \cdot \bar{z} = 0$

$\bar{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\bar{z} \cdot \bar{z} = \begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = 1 + i^2 = 0$

---

step 3:

If $\bar{z} = \bar{x} + i\bar{y}$ eigenvector of A w/ λ eigenvalue $\lambda \in \mathbb{C}$

then $\bar{z} \cdot \bar{z} = 0$ (previous) forces

$(\bar{x} + i\bar{y})(\bar{x} + i\bar{y}) = \underbrace{(|x|^2 - |y|^2)}_{=0} + \underbrace{(2\bar{x} \cdot \bar{y})}_{=0} i = 0 = 0 + 0i$

Exercise:

Let $T: E^2 \to E^2$ be the motion: reflect in x-axis, then rotate through angle θ around origin.

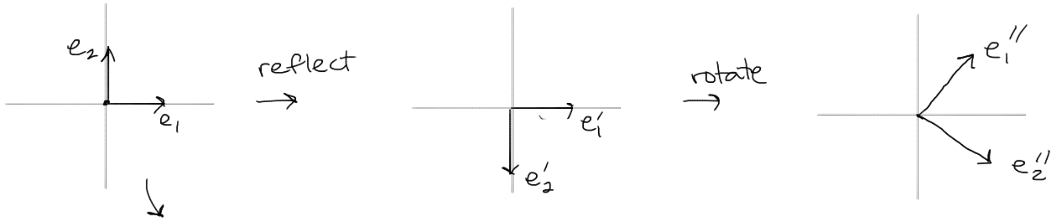
(a) Show T = reflection about a line, & describe the line.

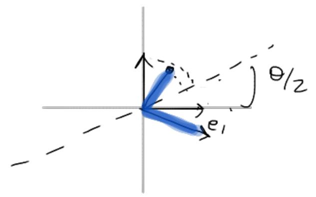
Know: $T(x) = Ax + \bar{b}$, $\bar{b} = \emptyset$ b/c no translation

$= Ax$

Rotation: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ Reflection about x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ matrix

$$T(x) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

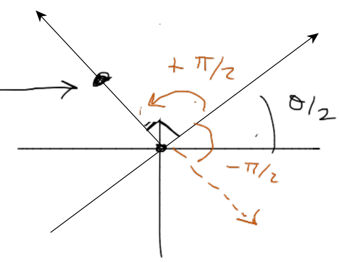




Line has slope $\tan(\theta/2)$

Line: $\theta/2$

Vector orthogonal to line $\begin{pmatrix} \cos(\frac{\pi}{2} + \frac{\theta}{2}) \\ \sin(\frac{\pi}{2} + \frac{\theta}{2}) \end{pmatrix}$



Compute

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{2} + \frac{\theta}{2}) \\ \sin(\frac{\pi}{2} + \frac{\theta}{2}) \end{pmatrix} = \begin{pmatrix} \cos\theta\cos(\frac{\pi+\theta}{2}) + \sin\theta\sin(\frac{\pi+\theta}{2}) \\ \sin\theta\cos(\frac{\pi+\theta}{2}) - \cos\theta\sin(\frac{\pi+\theta}{2}) \end{pmatrix} = \begin{pmatrix} \cos(\theta - (\frac{\pi+\theta}{2})) \\ \sin(\theta - (\frac{\pi+\theta}{2})) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\frac{\theta-\pi}{2}) \\ \sin(\frac{\theta-\pi}{2}) \end{pmatrix}$$

Next, find the eigenvectors of this matrix

Eign Decomp. $Ax = \lambda x \Rightarrow A\bar{x} - \lambda\bar{x} = 0 \Rightarrow A\bar{x} - \lambda I\bar{x} = 0 \Rightarrow (A - \lambda I)\bar{x} = 0$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \Rightarrow \det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{pmatrix} = -\cos^2\theta + \lambda^2 - \sin^2\theta = \lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\boxed{\lambda = \pm 1 \quad \text{Eign Values}}$$

$\lambda = 1 \Rightarrow$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

mult. bottom row × vector

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ 1 & \frac{-\cos\theta - 1}{\sin\theta} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad x - \frac{1 + \cos\theta}{\sin\theta} y = 0$$

$$x = \frac{1 + \cos\theta}{\sin\theta} y$$

$Ax = \lambda x$

$A(2x) = \lambda(2x)$

mult. by $\sin\frac{\theta}{2}$

Eigenvector

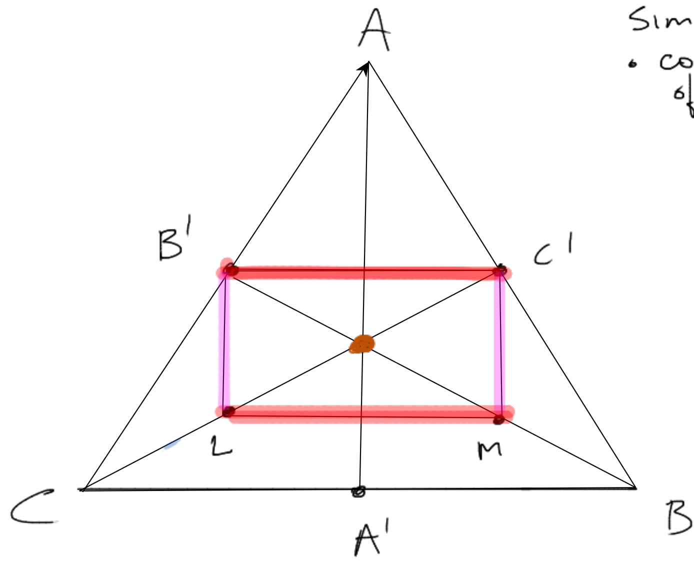
(scaling)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1 + \cos\theta}{\sin\theta} \cdot y \\ y \end{pmatrix} = y\begin{pmatrix} \frac{1 + \cos\theta}{\sin\theta} \\ 1 \end{pmatrix} = y\begin{pmatrix} \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\ 1 \end{pmatrix} = y\begin{pmatrix} \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \\ 1 \end{pmatrix} = y \cdot \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

Half-Angle: $\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta + 1}{2}} \Rightarrow 2\cos^2\frac{\theta}{2} = 1 + \cos\theta$

Double: $\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin(\theta) = \sin\left(2\cdot\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

P. 21



Similar Δ's:
- common angle & ratios of 2 sides the same

centroid

LM || ___ || ___ and LC' || ___ ← line has error.

should say

LM || A'C || B'C' and MC' || GA || LB'

LMA'C' parallel

LGC' is straight