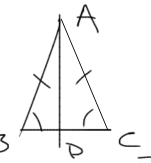
Spherical Geometry:

Pons Asinorum (Isosceles triangle theorem)

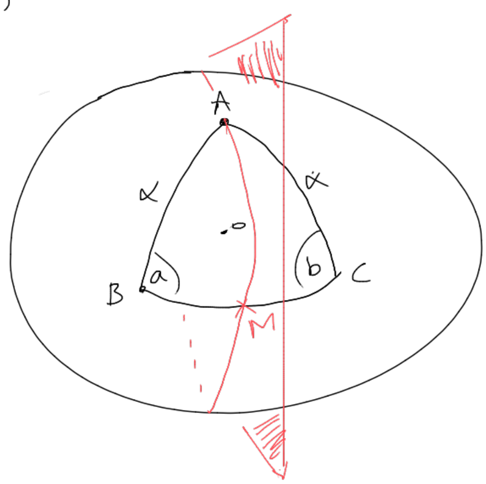
Recall, in Euclidean —



given $\overline{AB} = \overline{AC} \Rightarrow \angle C = \angle B$

proof via symmetries. reflection about AD is a motion, preserves angles takes BC to CB

Similarly in Spherical



Show angle $a = b$.

Motions of S^2 are a subset of the motions of $\mathbb{R}^3$.

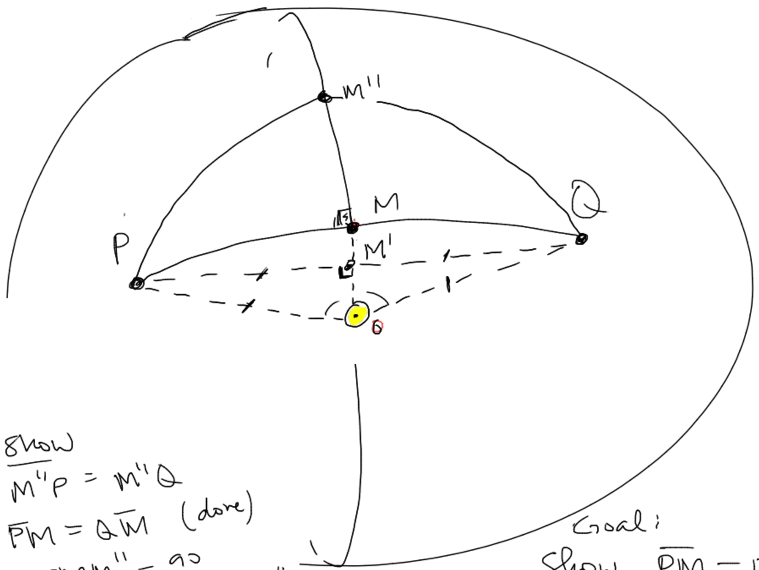
Reflection about plane containing $\{A, O, M\}$, w/ $M =$ midpt of BC

Motions preserve angles so $a = b$. b/c this reflection takes

$B \mapsto C$

$A \mapsto A$

Given, $P, Q \in S^2$ the set of points equidistant from both P, Q is the perp. bisector to segment PQ.



M = midpt of P, Q

($\exists$ plane $\perp$ to POQ thru M. Euclidean 3-D geom. argument)

- - - - lines are inside 3-D ball

Claim:

$\overline{PM'} = QM'$ (Plane that is the perp. bisector to PQ is the set of points whose dist to P is same as dist to Q and M' is in the plane)

$PO = OQ$ (radii)

$OM' = OM'$

SSS $\Rightarrow \triangle OPM' \cong \triangle OQM'$

$\Rightarrow \angle POM' = \angle QOM'$

Goal:

Show $\overline{PM} = \overline{QM}$ (spherical segments)

$\overline{QM}$ " angle b/w OQ and OM ($= \angle QOM'$)

$\overline{PM}$ || angle b/w $\overrightarrow{OP}$ and OM. ($= POM'$)

(equivalent from above)

Show

$M''P = M''Q$

$\overline{PM} = \overline{QM}$ (done)

$\angle PMM'' = 90 = \angle QMM''$

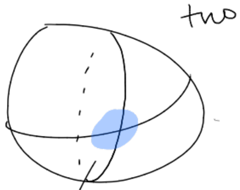
$MM'' = MM''$

SAS $\Rightarrow \overline{PM''} = \overline{QM''}$

Spherical vs. Euclidean Geometry

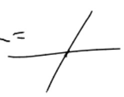
Similarities

1. triangle inequality
2. two pts. determine a line
3. local intersections



two sph. lines intersect exactly like two euclidean lines

zoom =



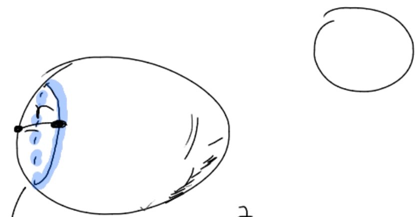
4. Distances (all things geometric) are defined by motions (preserved)
5. Homogeneity

Differences

1. Law of Cosines / Sines
2. Formulas for Area / Circumference of Circle

E: $A = \pi r^2$ | E: $C = 2\pi r$

Sp: $A = 2\pi(1-\cos(r))$ | Sp: $C = 2\pi \sin(r)$



S^2

$(x, y, z) \in \mathbb{R}^3$

choose coords s.t. $z = 0$

3. Bounded / Unboundedness of lengths / area
4. Triangle Angle Sum: $(\alpha+\beta+\gamma = \pi + \text{Area}\,\Delta)$

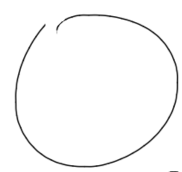
Sp: $\alpha+\beta+\gamma > \pi$

5. Frames

$\mathbb{E}^2$

any 2 ⊥ vectors determine a frame

Sp



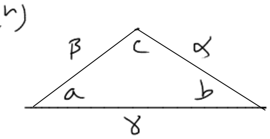
S^2 = unit sphere

(see text)

6 Intrinsic Curvature

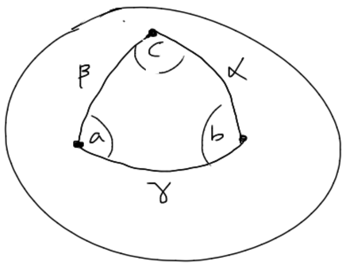
Spherical Law of Sines:

Recall ($\mathbb{E}^2$)



$$\frac{\sin a}{\alpha} = \frac{\sin b}{\beta} = \frac{\sin c}{\gamma}$$

Similarly for S^2:

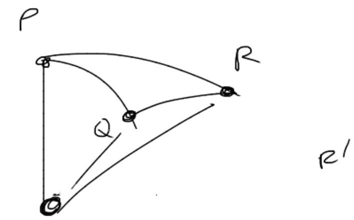


Choose coordinates as before:

$P = (0, 0, 1)$

$Q = (\sin\beta, 0, \cos\beta)$

$R = (\sin\gamma \cos a, \sin\gamma \sin a, \cos\gamma)$



Idea: compute det matrix w/ P, Q, R as columns.

Recall det is invariant to change of basis.

$$\left|\begin{pmatrix} 0 & \sin\beta & \sin\gamma\cos a \\ 0 & 0 & \sin\gamma \sin a \\ 1 & \cos\beta & \cos\gamma \end{pmatrix}\right| = \sin\beta \sin\gamma \sin a$$

$\sin b \sin c \sin \alpha$

Now choose different basis | Repeat

So, the 3 different bases are

$Q = (\sin\beta, 0, \cos\beta)$

$R = (\sin\gamma \cos a, \sin\gamma \sin a, \cos\gamma)$

$Q = (\sin\alpha, 0, \cos\alpha)$

$R = (\sin\beta \cos c, \sin\beta \sin c, \cos\beta)$

$Q = (\sin\gamma, 0, \cos\gamma)$

$R = (\sin\alpha \cos b, \sin\alpha \sin a, \cos\alpha)$