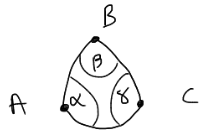
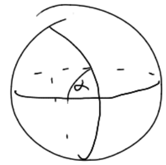
Area of spherical n-gons

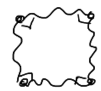
$n = 3$:

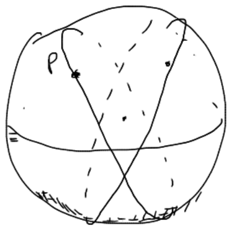


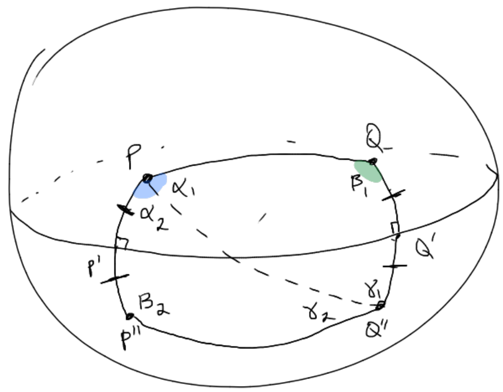


$$\text{Area} = \alpha + \beta + \gamma - \pi$$

Squares: do squares exist on a sphere?







Assumptions.

Let ℓ = a great circle

$d(P,\ell) = d(Q,\ell) = \frac{1}{2} d(P,Q)$

P' = pt. on ℓ closest to P

Q' = same for Q

Construct P'', Q''

Claim: $\square PQQ''P''$ is equilateral

Question: are all angles $= 90°$?

If so, let $\angle P'PQ = \angle PQQ' = 90$

Construct PQ'' diag

Label angles as shown: $\Rightarrow$ Area $\triangle PQQ'' = \alpha_1 + \beta_1 + \gamma_1 - \pi$

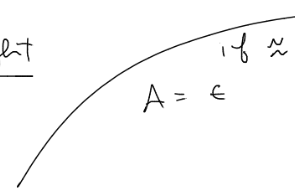
Area $\triangle PP'Q'' = \alpha_2 + \beta_2 + \gamma_2 - \pi$

$$\text{Area}\,\square\, PQQ''P'' = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) + (\gamma_1 + \gamma_2) - 2\pi$$

$$= \frac{\pi}{2} + \underbrace{\frac{\pi}{2} + \frac{\pi}{2}}_{\text{assume all angles are right}} + \frac{\pi}{2} - 2\pi$$

$\Rightarrow$ = 0

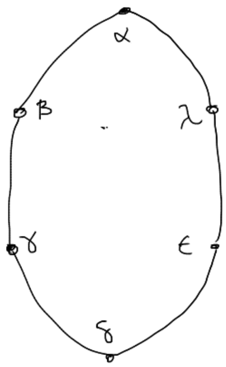
$\Rightarrow$ Squares don't exist on a sphere. equilateral equiangular = 90

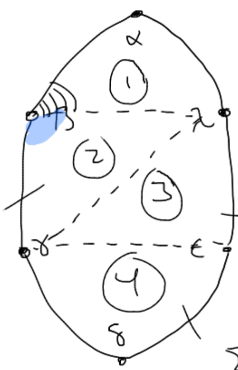


if $\approx \frac{\pi}{2} + \epsilon$

$A = \epsilon$

Spherical Hexagon





$\sum$ angles $= \pi$

$\sum$ angles $= \pi$

$\sum$ angles $= \pi$

$\sum$ angles $= \pi$

all angles are exactly angles of hexagon

$\Longrightarrow$

$$\boxed{\sum \text{angles} = 4\pi}$$

Do "similar triangles" exist on a sphere?

In Euclidean Geom: similar Δ's means: "same angles but different area"

— Gauss

In Sp. Geom: Area = sum of angles minus π

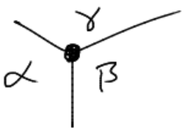
so same angles means same sum thus same area.

So we get a <u>new</u> congruence theorem

<u>Thm</u>: If two triangles have the same angles (i.e., they're similar) then they're congruent

Platonic Solids (regular) same length
same angle

① there are only 5, (know why)

if 

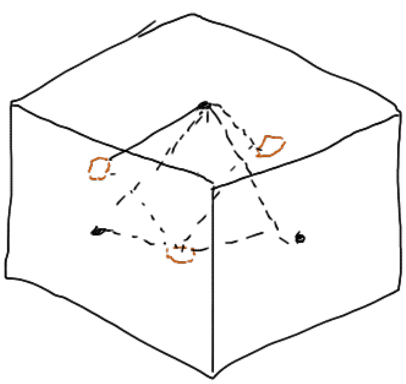
$\alpha + \beta + \gamma = 2\pi \Rightarrow$ the "graph" is planar

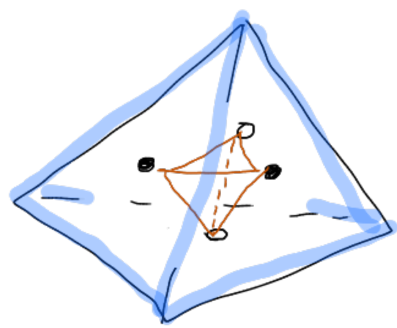
(same angles

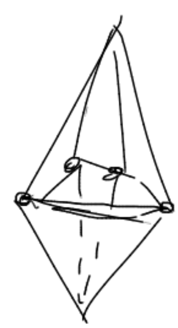
② duality

Dual of a Platonic solid:

- connect midpoints of adjacent faces.







③ Symmetries (motions)