1.7/ Let $T: E^2 \to E^2$ be motion: reflect in x-axis then rotate through θ around origin.
(a) Show T = reflection about a line & describe line.

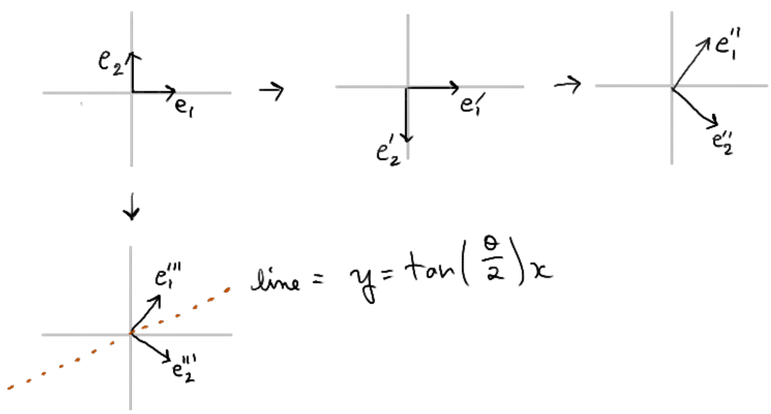
we know:
$$T(x) = T_1 \circ T_2(x)$$
$$T_2(\bar{x}) = A\bar{x} + \bar{b}$$

reflection $\Rightarrow$
$\bar{b} = \begin{pmatrix}0\\0\end{pmatrix}$, $A = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ | this works b/c $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix}$

So
$$T_2\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$T_1(\bar{x}) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}(\bar{x})$$

thus
$$T(x) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$



line $=$ $y = \tan\left(\frac{\theta}{2}\right)x$

CALCULATION

Line of Reflection: angle $\frac{\theta}{2}$

Vector orthogonal to Line: $\begin{pmatrix}\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\\ \sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\theta+\pi}{2}\right)\\ \sin\left(\frac{\theta+\pi}{2}\right)\end{pmatrix}$

Note:
$$\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi+\theta}{2}\right)\\ \sin\left(\frac{\pi+\theta}{2}\right)\end{pmatrix} =$$

$$\begin{pmatrix}\cos\theta\cos\left(\frac{\pi+\theta}{2}\right) + \sin\theta\sin\left(\frac{\pi+\theta}{2}\right)\\ \sin\theta\cos\left(\frac{\pi+\theta}{2}\right) - \cos\theta\sin\left(\frac{\pi+\theta}{2}\right)\end{pmatrix} =$$ almost trig sum formulas

$$\begin{pmatrix}\cos(-\theta)\cos\left(\frac{\pi+\theta}{2}\right) - \sin(-\theta)\sin\left(\frac{\pi+\theta}{2}\right)\\ \sin\theta\cos\left(\frac{-\pi-\theta}{2}\right) + \cos\theta\sin\left(\frac{-\pi-\theta}{2}\right)\end{pmatrix} =$$

$$\begin{pmatrix}\cos\left(-\theta + \frac{\pi+\theta}{2}\right)\\ \sin\left(\theta - \left(\frac{\pi+\theta}{2}\right)\right)\end{pmatrix} = \begin{pmatrix}\cos\left(\theta - \frac{\pi+\theta}{2}\right)\\ \sin\left(\theta - \frac{\pi+\theta}{2}\right)\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\theta-\pi}{2}\right)\\ \sin\left(\frac{\theta-\pi}{2}\right)\end{pmatrix}$$

(b) Eigen: $Ax = \lambda x \Rightarrow Ax - \lambda x = (A - \lambda I)x = 0$, so det must be 0

$$\det\begin{pmatrix}\cos\theta - \lambda & \sin\theta\\ \sin\theta & -\cos\theta - \lambda\end{pmatrix} = 0$$

$$-\cos^2\theta + \lambda^2 - \sin^2\theta = -1 + \lambda^2 = 0, \quad \lambda^2 = 1, \lambda = \pm 1$$

$\lambda = 1$:
$$\begin{pmatrix}\cos\theta - 1 & \sin\theta\\ \sin\theta & -\cos\theta - 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{matrix}0\\0\end{matrix}$$

$$\begin{pmatrix}\cos - 1 & \sin\theta\\ 1 & \frac{-\cos - 1}{\sin\theta}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow x - \left(\frac{\cos\theta + 1}{\sin\theta}\right)y = 0 \quad \text{or} \quad x = \left(\frac{\cos\theta + 1}{\sin\theta}\right)y$$

free var

factor scalar: $2\cos\frac{\theta}{2}$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{\cos\theta + 1}{\sin\theta}y\\ y\end{pmatrix} = \begin{pmatrix}(\cos\theta + 1)y\\ \sin\theta\, y\end{pmatrix} = y\begin{pmatrix}\cos\theta + 1\\ \sin\theta\end{pmatrix} = y\begin{pmatrix}\cos\theta + 1\\ \sin\theta\end{pmatrix} = y\begin{pmatrix}2\cos^2\left(\frac{\theta}{2}\right)\\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

Half-Angle: $\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta + 1}{2}} \Rightarrow 2\cos^2\frac{\theta}{2} = \cos\theta + 1$

Double-Angle: $\sin 2\theta = 2\cos\theta\sin\theta \Rightarrow \sin\frac{\theta}{2} = 2\sin\theta\cos\theta$

Symmetric matrix has $\perp$ e-vectors, so other is $\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix}$

1.8 (a) Let θ be a nonzero angle and $\mathbf{b}$ a translation vector in the plane. Give a geometric construction for a point $P \in \mathbb{E}^2$ such that

$$\mathrm{Rot}(O,\theta)(P) = \mathrm{Trans}(-\mathbf{b})(P).$$

[Hint: draw a picture, to find points P, Q with $\mathbf{b} = \overrightarrow{QP}$ such that O is on the perpendicular bisector of PQ and $\angle POQ = \theta$.]

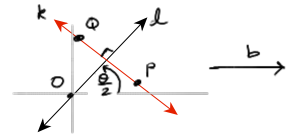
(b) By solving linear equations, find x, y such that

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

(c) Express the motion $T : \mathbb{E}^2 \to \mathbb{E}^2$ defined in coordinates by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in the form $T = \mathrm{Rot}(P, \theta)$.

(d) Relate (a) and (b).

(a) θ determines a unique line ℓ making angle $\frac{\theta}{2}$ w/ x-axis. Let k be line, $k \perp \ell$ and choose points $P, Q \in k$ s.t. $b = \overline{QP}$ and ℓ bisects $\overline{QP}$ and $\angle POQ = \theta$



TRIG FORMULAS

$2\cos^2\frac{\theta}{2} = \cos\theta + 1$

$2(\cos^2\frac{\theta}{2} - 1) = \cos\theta - 1$

$-2\sin^2\frac{\theta}{2} = \cos\theta - 1$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(b) Solve
$$\left.\begin{aligned} Ax + b &= \bar{x} \\ Ax - Ix &= -b \\ (A - I)\bar{x} &= -b \end{aligned}\right\} \quad \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$$

system of eq's

$$\left\{\begin{aligned} (\cos\theta - 1)x_1 + \sin\theta x_2 &= -b_1 \\ x_1 - \frac{(\cos\theta + 1)x_2}{\sin\theta} &= \frac{-b_2}{\sin\theta} \end{aligned}\right\} \Rightarrow x_1 = \frac{(\cos\theta + 1)x_2 - b_2}{\sin\theta} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{(\cos\theta + 1)x_2 - b_2}{\sin\theta} \\ x_2 \end{pmatrix}$$

$$= x_2\begin{pmatrix} \frac{\cos\theta + 1}{\sin\theta} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{-b_2}{\sin\theta} \\ 0 \end{pmatrix} \underset{\text{scale}}{=} \alpha\begin{pmatrix} \cos\theta + 1 \\ \sin\theta \end{pmatrix} + \begin{pmatrix} \frac{-b_2}{\sin\theta} \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} + \begin{pmatrix} \frac{-b_2}{\sin\theta} \\ 0 \end{pmatrix}$$

$$x_1 = \frac{-\sin\theta x_2 - b_1}{\cos\theta - 1} \quad \Bigg| \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta x_2/\cos\theta - 1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{-b_1}{\cos\theta - 1} \\ 0 \end{pmatrix} = x_2\begin{pmatrix} \frac{-\sin\theta}{\cos\theta - 1} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{-b_1}{\cos\theta - 1} \\ 0 \end{pmatrix}$$

(c) Let $T(x) = Ax + b$. then $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Part (b) suggests:

Let $\bar{p} = \begin{pmatrix} b_1 - b_2\cot(\theta/2) \\ b_1\cot(\theta/2) + b_2 \end{pmatrix}$. Now $T(x) = \mathrm{Rot}_{\bar{p}}(\bar{x})$

(d) The fixed point of (b) is fixed by the composition of the maps in (a)

$$(\cos\theta - 1)x_1 + \sin\theta x_2 + b_1 = 0$$
$$\sin\theta x_1 \quad (-\cos\theta - 1)x_2 + b_2 = 0$$

TRIG FORMULAS

$2\cos^2\frac{\theta}{2} = \cos\theta + 1$

$2(\cos^2\frac{\theta}{2} - 1) = \cos\theta - 1$

$-2\sin^2\frac{\theta}{2} = \cos\theta - 1$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}\begin{pmatrix} \bar{x} \end{pmatrix} = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$$

$$\bar{x} = \frac{1}{\underbrace{(\cos\theta - 1)^2 + \sin^2\theta}_{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}}\begin{pmatrix} -2\sin^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -2\sin^2\frac{\theta}{2} \end{pmatrix} = \frac{2\sin\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}}\begin{pmatrix} -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ -\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \end{pmatrix}$$

$$\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta = 2 - 2\cos\theta \qquad 2(1-\cos\theta) = 2(-2\sin^2\frac{\theta}{2}) = 4\sin^2\frac{\theta}{2}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \cot(\frac{\theta}{2}) \\ -\cot(\frac{\theta}{2}) & -1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$a - b\cot\frac{\theta}{2}$

$$\frac{1}{2}\begin{pmatrix} -b_1 + \cot\frac{\theta}{2} b_2 \\ -\cot\frac{\theta}{2} - b_2 \end{pmatrix}$$

$a\cot + b$

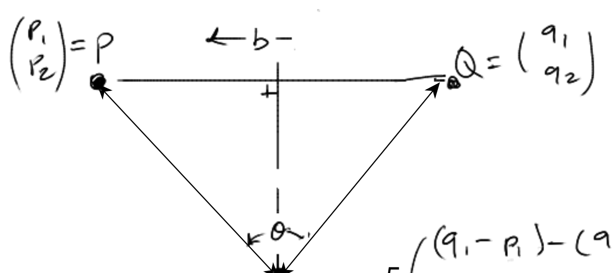
$$= \frac{1}{2}\begin{bmatrix} b_1 - b_2\cot(\frac{\theta}{2}) \\ b_1\cot\frac{\theta}{2} + b_2 \end{bmatrix} = C$$

this is the center of the rotation. this is C

point that gets fixed by
rotation through angle theta

$$\text{So} \quad T(x) = Ax + b = \mathrm{Rot}_C(\theta)$$

$T(x) = Ax + b$



$$C = .5\begin{pmatrix} (q_1 - p_1) - (q_2 - p_2)\cot(\frac{\theta}{2}) \\ (q_1 - p_1)\cot(\frac{\theta}{2}) + (q_2 - p_2) \end{pmatrix}$$

Q

1.8 (a) Let θ be a nonzero angle and $\mathbf{b}$ a translation vector in the plane. Give a geometric construction for a point $P \in \mathbb{E}^2$ such that

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[Hint: draw a picture, to find points P, Q with $\mathbf{b} = \overrightarrow{QP}$ such that O is on the perpendicular bisector of PQ and $\angle POQ = \theta$.]

(b) By solving linear equations, find x, y such that

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(d) Relate (a) and (b).

$Ax + b = x$

$(A - I)x = b$

$Ax = x - b$

$Aw = w - b$

−

$$\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

should be +

$$\begin{pmatrix}\cos\theta x_1 + \sin\theta x_2\\ \sin\theta x_1 - \cos\theta x_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}\cos\theta x_1 + \sin\theta x_2 + b_1\\ \sin\theta x_1 - \cos\theta x_2 + b_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$(\cos\theta - 1)x_1 + \sin\theta x_2 + b_1 = 0 \Rightarrow \sin\theta x_2 = -b_1 - (\cos\theta - 1)x_1$

$\sin\theta x_1 \quad (-\cos\theta - 1)x_2 + b_2 = 0 \qquad x_2 = \dfrac{-b_1 - (\cos\theta - 1)x_1}{\sin\theta}$

also $x_2 = \dfrac{b_2 + \sin\theta \cdot x_1}{(\cos\theta + 1)}$

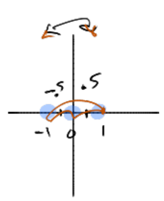
sol'n $\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\ \dfrac{-b_1 - (\cos\theta - 1)x_1}{\sin\theta}\end{pmatrix} = x_1\begin{pmatrix}1\\ \dfrac{-(\cos\theta - 1)}{\sin\theta}\end{pmatrix} + \begin{pmatrix}0\\ \dfrac{-b_1}{\sin\theta}\end{pmatrix}$

set $x_1 = 1$, then

$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\ f(\theta, b_1)\end{pmatrix}$

$b = \begin{pmatrix}2\\0\end{pmatrix}$

$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}($



$A = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

$\theta = \pi$

$b = \begin{matrix}1\\1\end{matrix}$

$Ax + b = x$

$(A - I)x = -b$

$\begin{pmatrix}-2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-b_1\\-b_2\end{pmatrix} \longrightarrow 0x + 0y = -b_2$

$0 = -b_2$

~~y free~~ no sol'n unless $-b_2 = 0$

$-2x = -b_1$

$x = \frac{1}{2} b_1$

$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}-x + 1\\ y + 1\end{pmatrix} = \begin{matrix}x\\y\end{matrix} \qquad 1 = 2x \qquad x = \frac{1}{2}$

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$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta x_1 - \sin\theta x_2 \\ \sin\theta x_1 + \cos\theta x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos\theta x_1 - \sin\theta x_2 + b_1 \\ \sin\theta x_1 + \cos\theta x_2 + b_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$(\cos\theta - 1)x_1 - \sin\theta x_2 + b_1 = 0 \Rightarrow \sin\theta x_2 = b_1 + (\cos\theta - 1)x_1$

$\sin\theta x_1 \quad (\cos\theta - 1)x_2 + b_2 = 0$

$$\boxed{x_2 = \frac{b_1 + (\cos\theta - 1)x_1}{\sin\theta}}$$

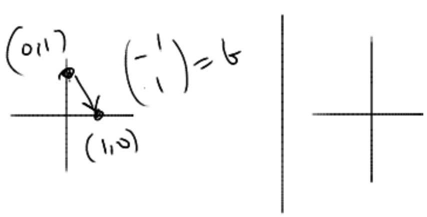
also

$$x_2 = \frac{b_2 + \sin\theta \cdot x_1}{1 - \cos\theta}$$

sol'n

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -\frac{b_1 - (\cos\theta - 1)x_1}{\sin\theta} \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -\frac{(\cos\theta - 1)}{\sin\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{b_1}{\sin\theta} \end{pmatrix}$$

$(\cos\theta + \sin\theta - 1)x_1 + (\cos\theta - \sin\theta - 1)x_2 + b_1 + b_2 = 0$



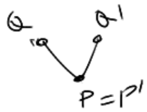
$x_1 =$

$Ax + b = \frac{1}{2}x$

$(A - \frac{1}{2})x$

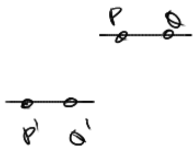
1.10

(a) $P = P'$



Set $\theta = \cos\left(\dfrac{(Q-P)\cdot(Q'-P)}{|Q-P||Q'-P|}\right)$ $\quad Rot_P(\theta)$ = rotation

(b) $PQ \parallel P'Q'$

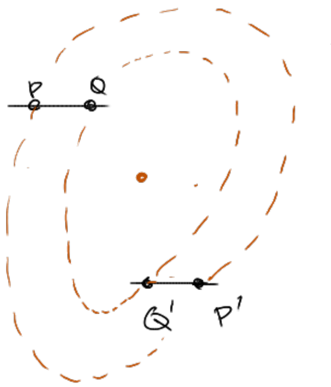


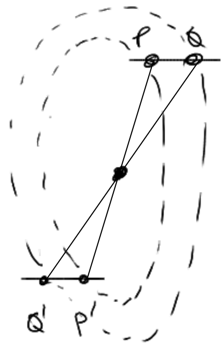
set $\bar{b} = P - P'$

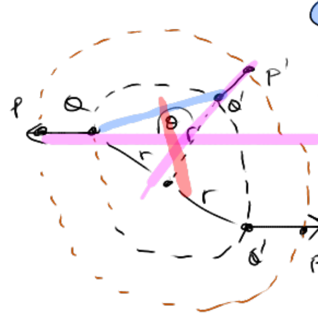
then $Trans_{\bar{b}}(P) = P'$

$Trans_{\bar{b}}(Q) = Q'$ (work in coords)

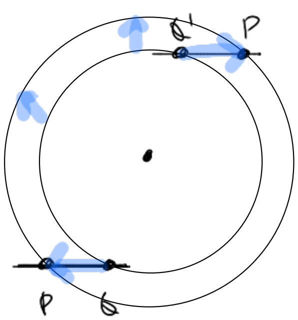
(c)







1. Form QQ' or PP' take midpoint
2. Take ⊥ bisector
3. Find point O on it s.t. $QOQ' = \theta$ where θ = angle b/w PQ, Q'P'



Set O = intersection of QQ' and PP'. 180 degree rotation about O will do it.

1.11 take $(0,0) \mapsto (1,2)$ & $(0,\sqrt{2}) \mapsto (2,3)$

Motion #1.

Lemma: To rotate about a point P by angle θ: conjugate rotation by translation

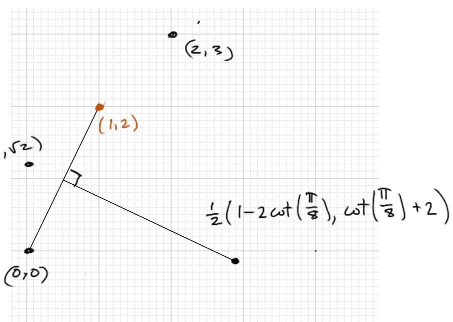
$\boxed{R_p(\theta) = T_p \circ R_0(\theta) \circ T_p^{-1}}$ so $\varphi(x) = A(\bar{x} - \bar{v}) + \bar{v}$

$\boxed{\varphi(x) = A\bar{x} - A\bar{v} + \bar{v}}$

so $\varphi(x) = A\bar{x} + \bar{b}$ w/ $\bar{b} = -A\bar{v} + \bar{v}$

1st translate by $\vec{v} = (1,2)$

$T_1(x) = x + \bar{v}$



2nd Compute angle b/w $(1, 2+\sqrt{2})$ & $(2,3)$, or equivalently, $(0,\sqrt{2})$ & $(1,1) = T_1^{-1}(2,3)$

$$\theta = \cos^{-1}\left(\frac{(0,\sqrt{2})\cdot(1,1)}{\sqrt{2}\cdot\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

notice negative ⇒ counter clockwise

3rd Follow w/ Rotate by $\theta = -\pi/4$ about the point $(1,2)$

$$x \longrightarrow x + v \xrightarrow{\text{Rotate by } \pi/4} \varphi(x) = A(\bar{x} - \bar{v}) + \bar{v}$$

$$\varphi \circ T_1(x) = \varphi(x + \bar{v}) = A(x + \bar{v} - \bar{v}) + \bar{v} = A\bar{x} + \bar{v}$$

b/c we've $(0,0)$: ← equivalent to rotate 1st then translate

$$\boxed{\varphi(x) = Ax + \bar{b} = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix}(\bar{x}) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

transformation: By (1.8) this is a rotation about P, w/ $A = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\text{Center} = \frac{1}{2}\begin{pmatrix} 1 - 2\cot(\pi/8) \\ \cot(\frac{\pi}{8}) + 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 - 2(-\sqrt{2}-1) \\ (-\sqrt{2}-1) + 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 + 2\sqrt{2} \\ 1 - \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 - 2(-\sqrt{2}-1) \\ (-\sqrt{2}-1) + 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + 2\sqrt{2} + 2 \\ 1 - \sqrt{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 + 2\sqrt{2} \\ 1 - \sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 2.9 \\ -0.2 \end{pmatrix}$$

So $\boxed{\varphi(x) = \mathrm{Rot}_p\left(\frac{\theta}{4}\right) \quad \text{where} \quad P = \frac{1}{2}\left(1 - 2\cot\left(\frac{\pi}{8}\right), \cot\left(\frac{\pi}{8}\right) + 2\right)}$

$\theta = \frac{\pi}{4} \Rightarrow$ Quad I ⇒ +

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos\theta}{2}}, \quad \text{so} \quad \cos\left(\frac{\theta}{8}\right) = \sqrt{\frac{1 + \sqrt{2}/2}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1 - \cos\theta}{2}}, \quad \text{so} \quad \sin\left(\frac{\theta}{8}\right) = \sqrt{\frac{1 - \sqrt{2}/2}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cot\left(\frac{\theta}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} \cdot \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \frac{2+\sqrt{2}}{\sqrt{4-2}} = \frac{2+\sqrt{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}+2}{2} = \sqrt{2} + 1$$

$$a^{1/2} b^{1/2} = (ab)^{1/2}$$

similarly $\cot\left(-\frac{\theta}{8}\right) = -(\sqrt{2}+1)$