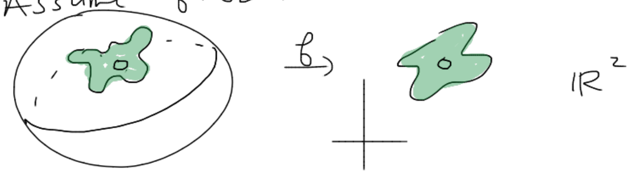
MA541 Wed

There is no isometry of a region in S^2 into $\mathbb{R}^2$.

Assume $f: \Omega \to \mathbb{R}^2$ is an isometry. Any such region contains some circle, C. w/ radius r.



The isometric image of a circle is a circle,

$\circledast$ (Isometries preserve circles)

$f(C) = C'$ w/ C' a circle in $\mathbb{R}^2$.

Know

circumference of $C = 2\pi \sin(r)$.

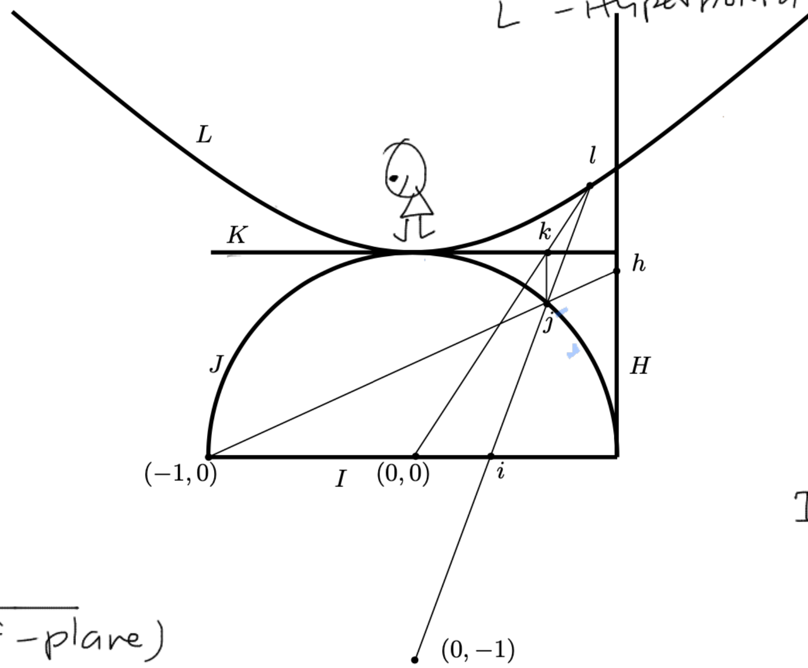
Circumf. of $C' = 2\pi\sin(r)$ b/c C' is isometric image of C.

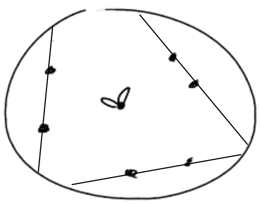
but C' is also a Euclidean circle $\circledast$. so circumf $= 2\pi r$

$\Rightarrow \sin(r) = r$

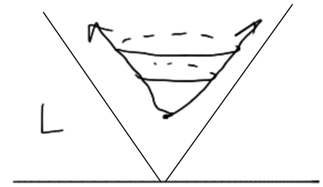
$\Rightarrow r = 0$ $\otimes$

H - half plane
I Poincare
J - semisphere
K - klein
L - Hyperboloid

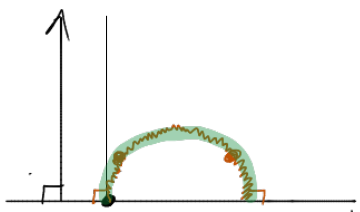




K



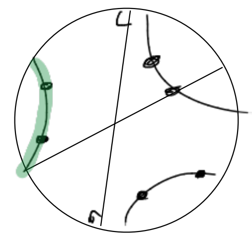
L



H (upper half-plane)

$x_{n+1} > 0$

I



geodesics on I

$$H = \{(1, x_2, \ldots, x_{n+1}) : x_{n+1} > 0\};$$
$$I = \{(x_1, \ldots, x_n, 0) : x_1^2 + \cdots + x_n^2 < 1\};$$
$$J = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1 \text{ and } x_{n+1} > 0\};$$
$$K = \{(x_1, \ldots, x_n, 1) : x_1^2 + \cdots + x_n^2 < 1\};$$
$$L = \{(x_1, \ldots, x_n, x_{n+1}) : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}.$$

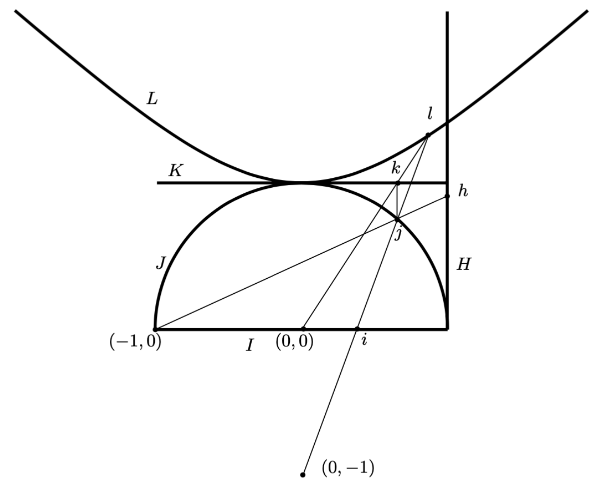


Figure 5. The five analytic models and their connecting isometries. The points $h \in H$, $i \in I$, $j \in J$, $k \in K$, and $l \in L$ can be thought of as the same point in (synthetic) hyperbolic space.

The map $\alpha : J \to H$ is central projection from the point $(-1, 0, \dots, 0)$:

$$\alpha : J \to H, \quad (x_1, \dots, x_{n+1}) \mapsto (1, 2x_2/(x_1+1), \dots, 2x_{n+1}/(x_1+1)).$$

The map $\beta : J \to I$ is central projection from $(0, \dots, 0, -1)$:

$$\beta : J \to I, \quad (x_1, \dots, x_{n+1}) \mapsto (x_1/(x_{n+1}+1), \dots, x_n/(x_{n+1}+1), 0).$$

The map $\gamma : K \to J$ is vertical projection:

$$\gamma : K \to J, \quad (x_1, \dots, x_n, 1) \mapsto \left(x_1, \dots, x_n, \sqrt{1 - x_1^2 - \cdots - x_n^2}\,\right).$$

The map $\delta : L \to J$ is central projection from $(0, \dots, 0, -1)$:

$$\delta : L \to J, \quad (x_1, \dots, x_{n+1}) \mapsto (x_1/x_{n+1}, \dots, x_n/x_{n+1}, 1/x_{n+1}).$$

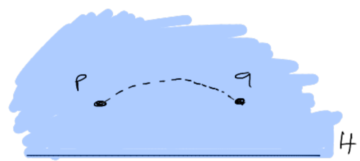
$$ds_H^2 = \frac{dx_2^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2};$$

$$ds_I^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)^2};$$

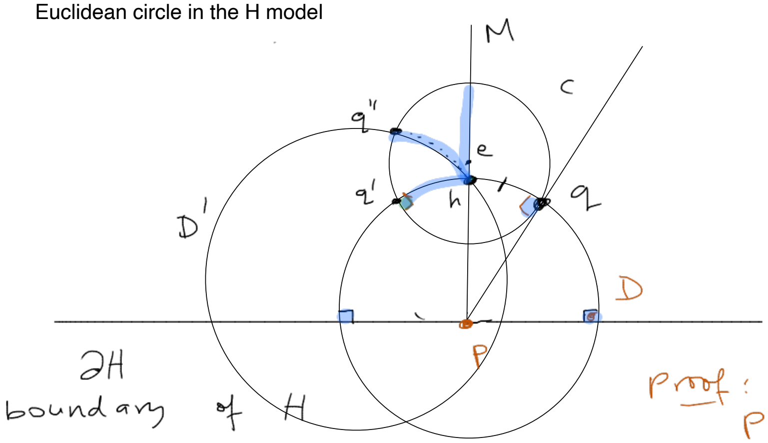
$$ds_J^2 = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2};$$

$$ds_K^2 = \frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)} + \frac{(x_1\,dx_1 + \cdots + x_n\,dx_n)^2}{(1 - x_1^2 - \cdots - x_n^2)^2};$$

$$ds_L^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$



Geometric construction for the hyperbolic center of a Euclidean circle in the H model



$e =$ Euclidean center of C

1. Draw vertical H-line M thru e, it hits ∂H @ p.
2. Draw tangent line to C that hits p, line $\cap\, C = q$
3. Draw circle D center $= p$, thru q
4. $D \cap M =$ hyperbolic center of C.

$\quad\quad\quad = h$

Proof: $p =$ center of D, $p \in \partial H =$ straight e-line

$\Rightarrow p \in$ diameter of D

$\Rightarrow D$ intersects diameter @ $90°$.

$\overleftrightarrow{pq}$ is diameter of D, tangent to C.

since $C \cap D = q$, C intersects D orthogonally,

$\underline{Q}$: $|eq| = |eq'|$

scaling is a hyperbolic isometry (Hyperbolic)

$f(\bar{x}) = 5\bar{x}$

horizontal translation is a hyperbolic (parabolic)

combine scaling + horiz translation to take $h \mapsto h$, $q' \mapsto q''$

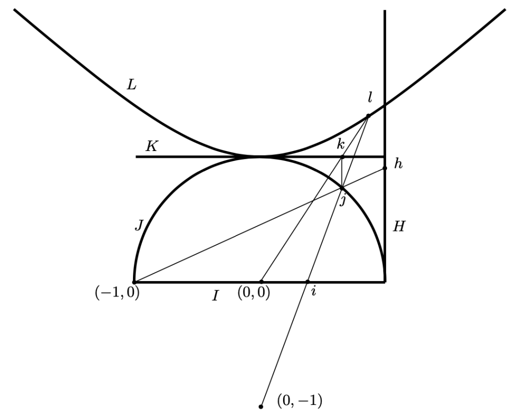


Figure 5. The five analytic models and their connecting isometries. The points $h \in H$, $i \in I$, $j \in J$, $k \in K$, and $l \in L$ can be thought of as the same point in (synthetic) hyperbolic space.

The map $\alpha : J \to H$ is central projection from the point $(-1, 0, \ldots, 0)$:

$$\alpha : J \to H, \quad (x_1, \ldots, x_{n+1}) \mapsto (1, 2x_2/(x_1+1), \ldots, 2x_{n+1}/(x_1+1)).$$

The map $\beta : J \to I$ is central projection from $(0, \ldots, 0, -1)$:

$$\beta : J \to I, \quad (x_1, \ldots, x_{n+1}) \mapsto (x_1/(x_{n+1}+1), \ldots, x_n/(x_{n+1}+1), 0).$$

The map $\gamma : K \to J$ is vertical projection:

$$\gamma : K \to J, \quad (x_1, \ldots, x_n, 1) \mapsto \left(x_1, \ldots, x_n, \sqrt{1 - x_1^2 - \cdots - x_n^2}\,\right).$$

The map $\delta : L \to J$ is central projection from $(0, \ldots, 0, -1)$:

$$\delta : L \to J, \quad (x_1, \ldots, x_{n+1}) \mapsto (x_1/x_{n+1}, \ldots, x_n/x_{n+1}, 1/x_{n+1}).$$

H, the Half-space model.

I, the Interior of the disk model.

J, the Jemisphere model (pronounce the J as in Spanish).

K, the Klein model.

L, the 'Loid model (short for hyperboloid).

$$\begin{aligned}
H &= \{(1, x_2, \ldots, x_{n+1}) : x_{n+1} > 0\};\\
I &= \{(x_1, \ldots, x_n, 0) : x_1^2 + \cdots + x_n^2 < 1\};\\
J &= \{(x_1, \ldots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1 \text{ and } x_{n+1} > 0\};\\
K &= \{(x_1, \ldots, x_n, 1) : x_1^2 + \cdots + x_n^2 < 1\};\\
L &= \{(x_1, \ldots, x_n, x_{n+1}) : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}.
\end{aligned}$$

$$\begin{aligned}
ds_H^2 &= \frac{dx_2^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2};\\
ds_I^2 &= 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)^2};\\
ds_J^2 &= \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2};\\
ds_K^2 &= \frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)} + \frac{(x_1\,dx_1 + \cdots + x_n\,dx_n)^2}{(1 - x_1^2 - \cdots - x_n^2)^2};\\
ds_L^2 &= dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.
\end{aligned}$$