Exam 1 (each problem is worth 100 points)

♯	1	2	3	4	5	6	7	8	Av
Points									

♯ 1: Find the explicit solution of the initial value problem and state the interval of existence.

$$\frac{dy}{dx} = \frac{x}{y(x^2-1)}, \quad y(0) = 1$$

equation is separable:

$$y\,dy = \frac{x}{x^2-1}\,dx$$

Integrate this: $\int y\,dy = \int \frac{x}{x^2-1}\,dx$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}\ln|x^2-1| + C$$

Invoke IC: $\frac{1}{2} = C \Rightarrow y^2 = \ln|x^2-1| + 1$

$$\Rightarrow \underline{y(x) = \sqrt{\ln(1-x^2)+1}} \text{ is solution of IVP } (x^2<1)$$

Find IoE: $\ln(1-x^2) > -1$

$$\Rightarrow 1-x^2 > e^{-1} \Rightarrow x^2 < 1 - 1/e$$

$$\Rightarrow \text{IoE: } -\sqrt{1-1/e} < x < \sqrt{1-1/e}$$

♯ 2: Consider the differential equation

$$\frac{dy}{dx} = \frac{xy}{x-1}. = f(x,y)$$

(a) Find the general solution.

(b) Does the equation admit a solution with initial condition $y(1) = 1$? If yes, find the solution. If not, explain why this does not contradict the basic theorem on the existence of solutions (see next page).

(a) separate: $\frac{dy}{y} = \frac{x\,dx}{x-1} = \left(1+\frac{1}{x-1}\right)dx$

integrate: $\int \frac{dy}{y} = \int \left(1+\frac{1}{x-1}\right)dx$

$\Rightarrow \ln|y| = x + \ln|x-1| + C_1$

$\Rightarrow |y| = \exp(C_1 + x + \ln|x-1|)$

$= e^{C_1} e^{x} e^{\ln|x-1|} = e^{C_1} e^{x} |x-1|$

general solution: $\underline{y(x) = De^{x}(x-1)}$, $D = \pm e^{C_1}$ or 0

(b) For any D, $y(1) = 0$, hence there is no solution that satisfies $y(1) = 1$

This does not contradict the basic existence theorem, because $f(x,y)$ is not continuous (not defined) at $(x,y) = (1,1)$.

♯ 3: (a) Find the general solution of the following equation:

$$\frac{dy}{dt} + y\cos t = \cos t$$

Use integrating factor: $u(t) = e^{\int \cos t\, dt} = e^{\sin t}$

Multiply equation by $e^{\sin t}$:

$$e^{\sin t}\frac{dy}{dt} + y e^{\sin t}\cos t = e^{\sin t}\cos t$$

$$\Rightarrow \frac{d}{dt}\left(y e^{\sin t}\right) = e^{\sin t}\cos t$$

Integrate: $y e^{\sin t} = \int e^{\sin t}\cos t\, dt = e^{\sin t} + C_1$

$\Rightarrow\ y(t) = 1 + C_1 e^{-\sin t}$ (general solution)

(b) Find the solution of the following initial value problem:

$$\frac{dy}{dt} = -2ty + 3t^2 e^{-t^2}, \quad y(0) = 1$$

Use variation of parameter: $y_h(t) = e^{\int(-2t)dt} = e^{-t^2}$

Particular solution: $y_p(t) = e^{-t^2}\int 3t^2 \frac{e^{-t^2}}{e^{-t^2}}\, dt = e^{-t^2} t^3$

General solution: $y(t) = C_1' e^{-t^2} + t^3 e^{-t^2}$

IC: $y(0) = C_1' = 1$

$\Rightarrow\ y(t) = e^{-t^2}(1+t^3)$ is solution of IVP

♯ 4: Find the general solution in implicit form for the equation below. *Hint:* The differential form associated with this equation is exact.

$$\frac{dy}{dx} = \frac{5x^4 - y^2}{2xy + 3y^2}$$

differential form: $\underbrace{(y^2-5x^4)}_{P(x,y)}dx + \underbrace{(2xy+3y^2)}_{Q(x,y)}dy = 0$

$P_y = 2y, \quad Q_x = 2y = P_y \Rightarrow$ form is exact.

Set $F(x,y) = \int P\,dx + \phi(y)$

$= \int (y^2-5x^4)\,dx + \phi(y)$

$= y^2x - x^5 + \phi(y)$

require $\frac{\partial F}{\partial y} = 2xy + \phi'(y) = Q = 2xy + 3y^2$

$\Rightarrow \phi' = 3y^2 \Rightarrow \phi(y) = y^3$

general solution in implicit form:

$F(x,y) = \underline{xy^2 - x^5 + y^3 = C}$

♯ 5: Find the general solution in implicit form for the equation below. *Hint:* The equation is homogeneous.

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

equation is homogeneous; set $y = xv$

Then $\frac{dy}{dx} = x\frac{dv}{dx} + v = \frac{1+3v^2}{2v}$

separate: $x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+v^2}{2v}$

$\Rightarrow \frac{2v}{1+v^2}\,dv = \frac{dx}{x}$

Integrate this: $\int \frac{2v}{1+v^2}\,dv = \int \frac{dx}{x}$

$\Rightarrow \ln(1+v^2) = \ln|x| + C_1 = \ln(e^{C_1}|x|$

$\Rightarrow 1+v^2 = e^{C_1}|x|$

$\Rightarrow 1+v^2 = Dx \quad (D = \pm e^{C_1}$ s.t. $Dx > 0)$

Substitute $v = \frac{y}{x} \Rightarrow 1 + \frac{y^2}{x^2} = Dx$

$\Rightarrow \underline{x^2+y^2 = Dx^3}$ is general implicit solution

♯ 6: Newton's law of cooling is given by

$$\frac{dT}{dt} = -k(T-R),$$

where $T(t)$ is the temperature at time t, R is the surrounding temperature, and k is a positive constant. Suppose a cold bottle of milk at 40^o F is placed into a warm room at 70^o F. Suppose 10 minutes later, the temperature of the milk is 50^o F. What is the temperature of the milk 20 minutes after the milk was placed into the room? Use the following table for your calculations. Note $\ln(x) = -\ln(1/x)$.

x	0.8	1	1.5
$\ln x$	-0.2	0	0.4
e^{-x}	0.4	0.35	0.2

Solution of DE: $T - R = (T_0 - R)e^{-kt}$ $\quad (T_0 = T(0))$

Hence $T(t) = R + (T_0 - R)e^{-kt}$

and $e^{kt} = \dfrac{T_0 - R}{T - R} \Rightarrow k = \dfrac{1}{t}\ln\left(\dfrac{T_0 - R}{T - R}\right)$

Data: $R = 70°F$

$T_0 = 40°F$

$T = 50°F$ at $t = 10$ min

$\Rightarrow k = \frac{1}{10}\ln\left(\frac{40-70}{50-70}\right) = \frac{1}{10}\ln(1.5) = \frac{1}{10}\cdot 0.4 = 0.04/\text{min}$

At $t = 20$ min: $T(20) = 70 + (40-70)e^{-0.04\cdot 20}$

$= 70 - 30e^{-0.8}$

$= 70 - 30\cdot 0.4$

$= 70 - 12$

$\underline{T(20) = 58°F}$

♯ 7: Consider the autonomous equation

$$\frac{dx}{dt} = (x+2)(x+1)(x-1)(x-2) = x^4 - 5x^2 + 4. = f(x)$$

Find all equilibrium points, classify their stability, and sketch the phase line diagram. Use this information to sketch the solution curves in the (t, x)-plane ($t \geq 0$) for the initial conditions $x(0) = -2$, $x(0) = -1.5$, $x(0) = 0$, $x(0) = 1$.

EPs: $x_e = -2, -1, 1, 2$

Stability: $f'(x) = 4x^3 - 10x$

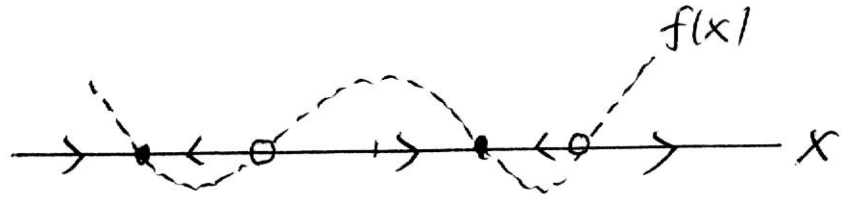
$f'(-2) = -32 + 20 < 0 \Rightarrow -2$ is as. stable

$f'(-1) = -4 + 10 > 0 \Rightarrow -1$ is unstable

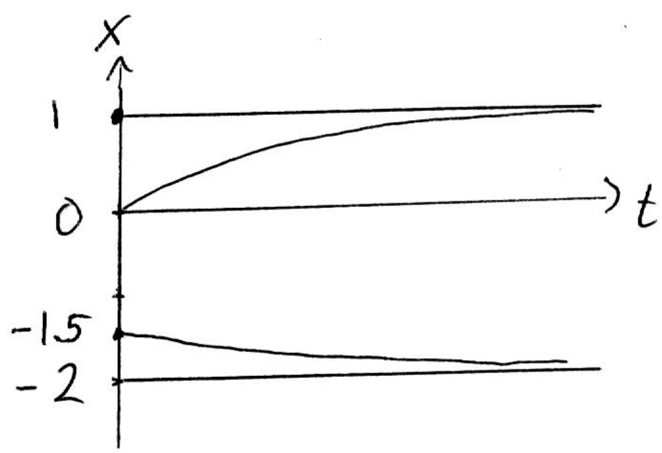
$f'(1) = 4 - 10 < 0 \Rightarrow 1$ is as. stable

$f'(2) = 32 - 20 > 0 \Rightarrow 2$ is unstable.

Phase line diagram:



Solution curves:



♯ 8: Define a matrix A, and vectors $\mathbf{x}$ and $\mathbf{b}$, and express the system of equations below in matrix-vector form $A\mathbf{x} = \mathbf{b}$. Use the augmented matrix to solve for $\mathbf{x}$ and express your answer in parametric form if the solution is not unique.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ x_2 - x_3 &= 3 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= 10 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 5 \\ 3 \\ 10 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

equation system: $A\underline{x} = \underline{b}$

Augmented matrix:

$$M = [A, \underline{b}] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 0 & 3 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow (2)$$

(1): $row_3 \to row_3 - 2row_1$

(2): $row_1 \to row_1 - 2row_2$

$$RREF(M) = \begin{bmatrix} 1 & 0 & 5 & 4 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot variables: x_1, x_2 free variables: $x_3 = s, x_4 = t$

transformed equations from RREF(M): $x_1 = -5s - 4t - 1$

$x_2 = s + 3$

Solution in parametric form:

$$\underline{x} = \begin{bmatrix} 5s - 4t - 1 \\ s + 3 \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$