Name: KEY

Second midterm practice exam, Math 340

Each problem is worth 10 points. One problem with the lowest score will be dropped. Good luck!

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total/Total - lowest	

1. (10 pts) Find the basis for the nullspace of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & -3 \end{pmatrix}$$

The question is equivalent to solving $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$. Solving this system is done by finding RREF or REF of A.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & -3 \end{pmatrix} \xrightarrow[\substack{\& \\ -R_3+R_4 \to R_4}]{-2\text{Row}1+\text{Row}2 \to \text{Row}2} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-R_2+R_3 \to R_3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Row1 + Row3 → Row3

This is equivalent to:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_4 = t$ (x_4 is free)

$x_3 = s$ (x_3 is too)

$x_2 - 3x_4 = 0 \Rightarrow x_2 = 3t$

$x_1 + x_3 + x_4 \Rightarrow x_1 + s + t = 0 \Rightarrow x_1 = -s - t$

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - t \\ 3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = sv_1 + tv_2$

$\Rightarrow$ a basis for the nullspace is $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

2. (10 pts) Find all values of x for which the matrix $A(x)$ is singular. For these values of x, find the basis of the null space.

$$A(x) = \begin{pmatrix} 1 & x & x \\ x & 0 & 1 \\ 1 & x & -2 \end{pmatrix}$$

$A(x)$ is singular $\Leftrightarrow$ $\det(A) = 0$

$$\Leftrightarrow -x\begin{vmatrix} 1 & x \\ 1 & -2 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & x \\ 1 & x \end{vmatrix} = 0$$

$$\Rightarrow -x(-2-x) + 0 - (x-x) = 0$$

$$-x(-2-x) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = -2$$

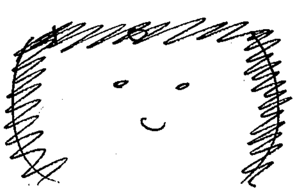
Case I: $x = 0$

$$A(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{2R_2 + R_3 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_3 = 0 \\ x_2 = t \quad (x_2 \text{ is free}) \\ x_1 = 0 \end{array}$$

so, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, a basis for Nul(A) is: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Case II: $x = -2$

$$A(-2) = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 1 \\ 1 & -2 & -2 \end{pmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{2R_1 + R_2 \to R_2} \begin{pmatrix} 1 & -2 & -2 \\ 0 & -4 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$



$\Rightarrow$ $x_3 = t$

$-4x_2 = 3t \Rightarrow x_2 = -\frac{3}{4}t$

$x_1 - 2x_2 - 2x_3 = 0$

$\Rightarrow x_1 + \frac{6}{4}t - 2t = x_1 - \frac{2}{4}t = 0$

$\Rightarrow x_1 = \frac{1}{2}t$

so: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ -\frac{3}{4}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$

so a basis for Nul(A) is $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$

(you should check that $A\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 0$)

3. (10 pts) Find all values of b for which the system

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + x_2 + 2x_3 = 3 \\ -x_1 - 2x_2 + 5x_3 = b \end{cases}$$

has a solution. Find that solution, and write it in the parametric form if necessary.

System is equivalent to: $\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 3 \\ -1 & -2 & 5 & b \end{pmatrix}$

$\xrightarrow{R_1 + R_3 \to R_3} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & -1 & 4 & 1+b \end{pmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & -1 & 4 & 1+b \end{pmatrix}$

$\xrightarrow{-R_2 + R_3 \to R_3} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & 0 & b \end{pmatrix} \Rightarrow b = 0.$ (otherwise systems is inconsistent)

Then $\underline{x_3 = t}$ since x_3 is free $(0x_1 + 0x_2 + 0x_3 = 0)$

(Row 2) $\Leftrightarrow$ $-x_2 + 4t = 1 \Rightarrow -x_2 = 1 - 4t \Rightarrow \underline{x_2 = 4t - 1}$

(Row 1) $\Leftrightarrow$ $x_1 + x_2 - \overset{"t"}{x_3} = x_1 + 4t - 1 - t = 1$

$\Rightarrow x_1 + 3t = 2 \Rightarrow x_1 = 2 - 3t$

$$\boxed{\text{So: } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 3t \\ 4t - 1 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}}$$

4. (10 pts) Write the initial value problem as a system of first-order equations using vector notation.

$$y'' + 2y' + 4y = 3\cos(wt), y(0) = 1, y'(0) = 0$$

Goal: $x' = Ax + f$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$

$x_1 = y \Rightarrow x_1' = x_2$

$x_2 = y' \Rightarrow x_2' = y'' = 3\cos(\omega t) - 2y' - 4y = 3\cos(\omega t) - 2x_2 - 4x_1$

So

$$\underline{\underline{\begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3\cos(\omega t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'}}$$

(this has form $Ax + f = x'$)

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ @ $t=0$ is $\underline{\underline{x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}}$

Done.

5. (10 pts) Consider the system

$$x_1' = x_1 x_2,$$
$$x_2' = x_2.$$

(a) Show, by direct substituition, that

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are solutions of the system.

(b) Show that $\mathbf{x}(\mathbf{t}) + \mathbf{y}(\mathbf{t})$ is **not** a solution of the system. Why doesn't this contradict the theorem of superposition?

(a) $x_1 = 0$, $x_2 = e^t \Rightarrow$ $x_1' = 0 = 0 \cdot e^t = x_1 \cdot x_2$ ✓
$x_2' = e^t = x_2$

$y_1 = 1$, $y_2 = 0 \Rightarrow$ $y_1' = 0 = 1 \cdot 0 = y_1 \cdot y_2$ ✓
$y_2' = 0 = y_2$ ✓

(b) $x(t) + y(t) = \begin{bmatrix} 1 \\ e^t \end{bmatrix}$; $[x(t) + y(t)]' = \begin{bmatrix} 1 \\ e^t \end{bmatrix}' \doteq \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$

$z_1' = 0 \neq z_1 \cdot z_2 = 1 \cdot e^t$ (this is enough to show $x(t) + y(t)$ is not a sol'n.)

$z_2' = e^t = z_2$ (doesn't matter)

The given ODE is not linear, so the theorem of superposition does not apply.

6. (10 pts) Find a fundamental set of solutions for the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ where A is the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \Rightarrow p(\lambda) = (-1-\lambda)^2$$

$$\Rightarrow \lambda = -1 \quad m_a = 2$$

Since $A - (-I) = A + I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A - \lambda I$ has $m_g = 2$

$\Rightarrow$ every vector is eigenvector.

So if we choose $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as eigenvectors

$x(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the general sol'n,

& so $x_1(t) = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

&

$x_2(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are a fundamental set of solutions

7. (10 pts) Find the solution to the initial-value problem for the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with the matrix A and initial value given below.

$$A = \begin{pmatrix} 2 & -6 \\ 0 & -1 \end{pmatrix} \text{ and } \mathbf{y}(\mathbf{0}) = (\mathbf{0}, \mathbf{1})^{\mathrm{T}}$$

$$A - \lambda I = \begin{pmatrix} 2-\lambda & -6 \\ 0 & -1-\lambda \end{pmatrix} \Rightarrow (2-\lambda)(-1-\lambda) = 0$$

$$\Leftrightarrow (2-\lambda)(\lambda+1) = 0$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = -1$$

real, distinct so —

gen'l sol'n is:

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{2t} v_1 + c_2 e^{-t} v_2$$

where v_i is eigenvector for λ_i.

$\underline{\lambda_1 = 2}$: $A - 2I = \begin{pmatrix} 0 & -6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} w_1 = t \\ w_2 = 0 \end{matrix}$ & $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is eig. vector

$\lambda_1 = -1$ $A + I = \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3u_1 - 6u_2 = 0$

$\Rightarrow 3u_1 = 6u_2 \Rightarrow \underline{u_1 = 2u_2}$

so $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigenvector

Thus $y(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

& $y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ c_2 \end{bmatrix} \Rightarrow \underline{c_2 = 1}$

& $0 = c_1 + 2 \Rightarrow \underline{c_1 = -2}$

Sol: $y(t) = -2e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$