Math 340

Notes on Section 1.1

Some differential equation models:

Population, Mechanics & Pollution

Population: World population in 2004 = 6.988 billion

(i) common assumption: $\frac{dP}{dt}$ is proportional to P.

rate of change is (sometimes) proportional to quantity

If $\frac{dP}{dt} = \lambda P$, for some λ (growth rate)

(note: Mexico @ 2004 $\Rightarrow \lambda = 1.2\%$
Sweden @ 2004 $\Rightarrow \lambda = .2\%$)

we may $\boxed{\text{seperate variables}}$

So, $\frac{dP}{P} = \lambda\, dt$ & then $\int$ both sides

$$\int \frac{dP}{P} = \int \lambda\, dt$$

$\ln|P| = \lambda t + C$ then exp both sides

$$\boxed{P = C_0 \cdot e^{\lambda t}}$$

Exponential Model of Population

TREAT $\frac{dP}{dt}$ "the derivative"

AS THE RATIO OF DIFFERENTIALS

...

Mathematically sound since:

$\frac{dP}{dt} = P' \Rightarrow dP = P' \cdot dt$

(derivative of P wrt t) multiplied by an ∞-small change in t gives the ∞-change in P

TEMPLATES FOR MOTION IN GREAT GENERALITY

FROM MECHANICS
(1670's)

Newton's 2nd law: Force = RATE OF CHANGE OF MOMENTUM

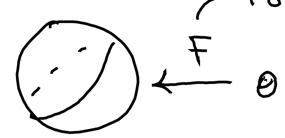
Momentum = mass × velocity

so $\frac{d}{dt}(mv) = m \cdot \frac{dv}{dt} = m \cdot a$

acceleration

Newton's Law of Gravitation:

Force w/ magnitude $|F| = \frac{G \cdot M \cdot m}{r^2}$

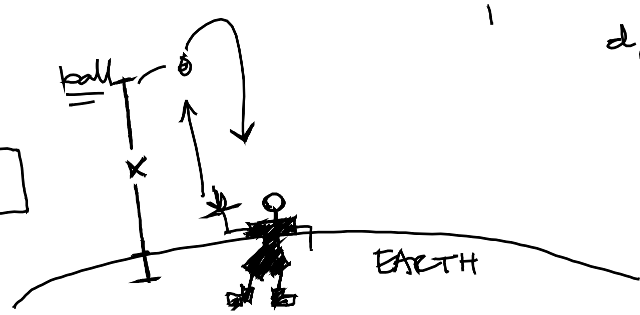


mass M

mass m

d distance

Simple Example



x = distance of ball to earth

$v = \frac{dx}{dt}$ velocity

$a = \frac{dv}{dt} = \frac{d^2x}{dt}$ acceleration

since r is so small relative to size (diameter) of the earth $g = \frac{G \cdot M}{r^2}$ (constant)

Earth's acceleration due to gravity

So, the $|F|$, above may be written:

$F = -g \cdot m$ (mass × acceleration)

negative force decreases x (positive):

FINALLY $-mg = ma$

$= m\frac{dv}{dt}$

$= m \cdot \frac{d^2x}{dt^2}$

(m's cancel)

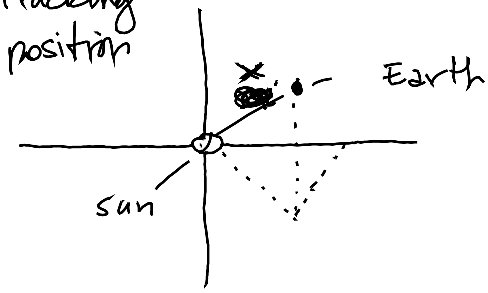
DIFFERENTIAL EQUATION MODELING MOTION OF BALL

$$\boxed{\frac{d^2x}{dt^2} = -g}$$

BUT THERE'S MORE ...

... Imagine the sun fixed at the origin

$x(t)$: vector tracking Earth's position



$$x(t) = \begin{bmatrix} w \\ y \\ z \end{bmatrix}$$

(3 coordinates)

Derivative : $$v(t) = \frac{d\vec{x}}{dt} = \begin{bmatrix} dw/dt \\ dy/dt \\ dz/dt \end{bmatrix}$$

$F = ma \Rightarrow$ $$F = m\frac{d^2\vec{x}}{dt^2} = -m \cdot \frac{GM}{|\vec{x}|^2} \cdot \frac{\vec{x}}{|\vec{x}|}$$

mass

acceleration

direction

This vector equation hides 3 differential equations (2nd order)

& these are Newton's Model of planetary motion

ALL ABOUT THE DERIVATIVE

Recall, there are many different (but equivalent) ways to think about the derivative.

1. Rate of Change of a function..

modeling

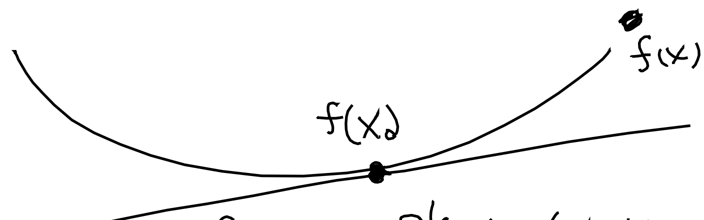
Q. What functions can you describe so that the derivative is meaningful.

Q What variables may be present & how does the derivative respect this?

Ex
- Position, Velocity, Acceleration
- Light absorbed as function of surface area

2. SLOPE of Tangent Line

geometry



$$y = f(x_0) + f'(x)\cdot(x - x_0)$$

3 Best Linear Approximation

$$L(x) = f(x_0) + f'(x)(x - x_0)$$

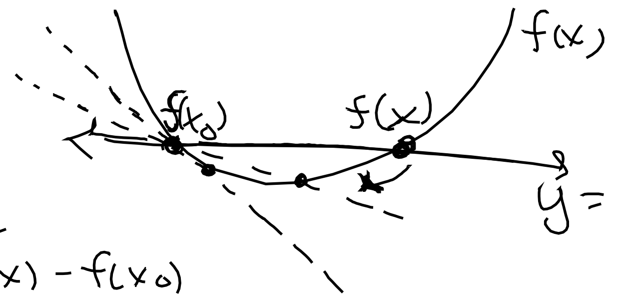
Taylor's Thm: $f(x) = \underbrace{f(0) + x\cdot f'(0)}_{\text{Basically } L(x)} + \underbrace{x^2 \cdot f''(0) + \ldots}_{\text{NON-LINEAR}}$

$\Rightarrow f(x) = L(x) + R(x)$ &

$R(x) \to 0$ as $x \to x_0$

(MORE INTERPRETATIONS OF THE DERIVATIVE)

$\boxed{4}$ The limit of difference quotients



$$y = f(x_0) + m(x - x_0)$$

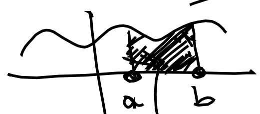
slope

$$m = \frac{f(x) - f(x_0)}{x - x_0}$$

As $x \to x_0$, the secant line approaches the tangent line

INTEGRATION

Two major perspectives: Area under curve & Anti-derivatives



$\int_a^b f(x)\,dx$

Depending on the context, or physical meaning of $f(x)$,

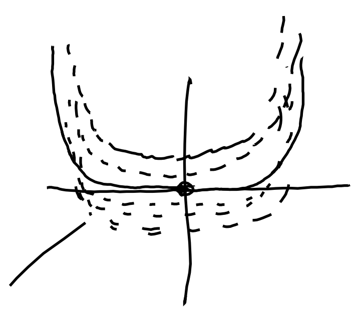
the area under the curve can mean different things - like: distance (if $f(x)$ is velocity).

MANY DIFFERENTIAL EQUATIONS CAN BE SOLVED BY DIRECT INTEGRATION

Even for multi-variable functions (when all but one variable is held constant), direct integration is useful.

Ex: $y' = x^3 \Rightarrow y = \int x^3\,dx = \frac{x^4}{4} + C$

↳ constant of integration



[vertical Translates of each other]

If we are given add'l info: $y(3) = 5$, we call this an initial value problem

Examples of general solution

Ex $\quad y' = xe^x \; ; \quad y(1) = 3.$

when integrating a product, often help to integrate by parts.

$\boxed{\text{IBP}}$ If you forget the formula — derive it.

(From Product Rule) $\quad [f \cdot g]' = f' \cdot g + g' \cdot f$

$$\Rightarrow [f \cdot g]' - g' \cdot f = f' \cdot g$$

$$\Rightarrow \int [f \cdot g]' dx - \int g' \cdot f \, dx = \int f' \cdot g \, dx$$

$$\left[\begin{array}{c} u \cdot v - \int u \cdot v' = \int u' \cdot v \\ \text{or} \\ v \cdot u - \int v \cdot u' = \int u \cdot v' \end{array}\right] \Leftarrow\Rightarrow \boxed{f \cdot g - \int g' \cdot f \, dx = \int f' \cdot g \, dx}$$

by swapping roles of f & g.

In this example, $\quad u = x, \; dv = e^x$
$\quad du = dx, \; v = e^x.$

so $\quad y = \int y' = \int xe^x = \int u \, dv = v \cdot u - \int v \, du$

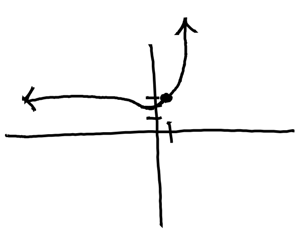
$$= x \cdot e^x - \int e^x dx = xe^x - e^x + c$$

since $y(1) = 3 \Rightarrow \quad 3 = 1 \cdot e^1 - e^1 + c$
$\quad 3 = c$

so: $\quad y = e^x(x-1) + 3$

Let's Check: $\quad y' = e^x(1) + 0 + e^x(x-1) + 0$

$$= e^x \cdot x \quad \checkmark$$



see fooplot.com

or create a table of values

NOTE: This sol'n is the member of family which passes through $(1,3)$