Abstract. We introduce a type of surgery on metric spaces. This surgery, in some sense, seeks to replace a subspace S of a metric space X with another metric space T via a function $f\colon S \to T$. When T is a discrete space, this amounts to collapsing the subspace according to the function. This surgery results in a new metric space we denote $\widehat{X}_f$ and there is a natural function $F\colon X \to \widehat{X}_f$ induced from f. Our primary interest is investigating if properties of the original function f are inherited by the induced function F. We show that if f is a pseudo-isometry then so is F. However, for a quasi-isometry, a very natural generalization of a pseudo-isometry that is prevalent in geometric group theory, such a result does not hold.

Pseudo-Isometric Surgery

Double Blind No Name1 and Double Blind No Name2

1. INTRODUCTION The idea of removing a subset from a space and replacing it with a modified version is one the most basic transformations of mathematics. Its roots stretch back to aniquity where the ambiguous term ‘superposition’ was occasionally used by Euclid. The Möbius band is an early example of this kind of *cut and paste* operation. Indeed, removing a rectangle from the annulus and pasting in a twisted version results in a surface fundamentally different than that of the original annulus, see Figure 1. Such transformations are used to produce new spaces that are simulatneously different from, yet similar to, the original.

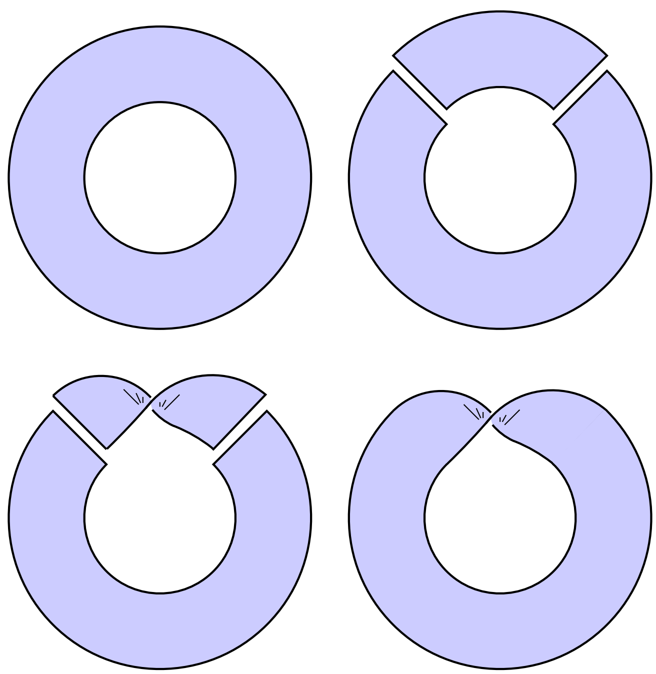

Figure 1. The Möbius band is obtained from a surgery on the annulus.

In 1910, some fifty years after Möbius and Listing described properties of the Möbius band, Max Dehn introduced a type of surgery in three dimensions. It took another fifty years for the concept of surgery to become widely used, appearing in works by Milnor and Thom [1]. Dehn's operation, now known as *Dehn Surgery*, is where one first removes a solid torus T from a 3–manifold and then 'sews it back differently', see [2] and [3] for details. There are many different ways to sew the solid torus back in. Specifically, note that simple closed curves on the torus can be identified with ordered pairs of relatively prime integers (m, n) corresponding to how the curve winds around the surface. Once the solid torus T is removed we can glue it back in so that the (m, n) curve is sewn to the curve that previously matched the $(1, 0)$ curve. Each different choice of m and n results in a potentially different 3–manifold. In fact, every closed, orientable, connected 3–manifold can be obtained by Dehn surgery on a collection of solid tori in the 3-sphere; a result known as the Lickorish–Wallace theorem [4]. Here we see that surgery on a localized subset can not only greatly alter the global structure of the original space, but also produce new spaces.

One might also examine the impact of surgery on the underlying geometry of a space. Consider $\mathbb{R}$ under the standard metric. For this surgery, instead of 'sewing it back differently' we 'sew in something else' as follows: remove every interval of the form $[2n, 2n + 1]$ where $n \in \mathbb{Z}$ and replace them with the points $2n$. Under the standard metric, $\mathbb{R}$ is a path metric space. In other words, if the distance between any two points in $\mathbb{R}$ is δ, then there is a path in $\mathbb{R}$ whose length is (perhaps approximately) δ. The same is true for the surgered space Y and it is not hard to convince yourself that the resulting space Y is isomorphic to $\mathbb{R}$. Essentially by construction, the length of the shortest path between a point in Y to some other point in Y will be distorted from what it was before the surgery. Said in another way, the surgery that collapses these closed intervals to their left endpoints distorts distances. This surgery is clearly not an isometry but, as will become clear later, it is a *quasi-isometry* (in fact also a pseudo-isometry too).

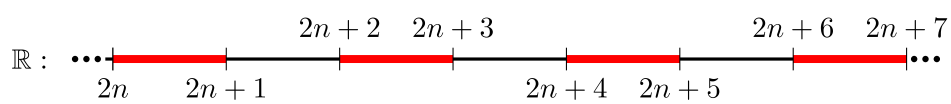

Figure 2. Replacing the red sets by points gives a space quasi-isometric to $\mathbb{R}$.

We are interested in surgery of an arbitrary metric space, one in which in notion of path may not exist. Since the metric in the surgered space cannot be defined in terms of paths, we use a discrete approximation of a path—a sequence of points. The aim of this paper is to attempt to quantify how much distortion can result from a particular type of surgery.

One very important class of functions on metric spaces that allow for a controlled distortion is a *quasi-isometry*. Such a function is a transformation between metric spaces that distorts distances by a uniformly bounded amount, above a given scale. A precise definition is given in Definition 2.1 and its subsequent remark. Implicit in the work of Švarc [5] in 1955 and Milnor [6] in 1968 the notion of a quasi-isometry is central to Gromov's idea of *coarse equivalence* of metric spaces. In 1981, Gromov [7] defined quasi-isometry the way it is used today. A standard example of a quasi-isometry is the (discontinuous) map that sends each real number to the greatest integer less than or equal to it [8]. For a general introduction to quasi-isometries, see [9].

Not all quasi-isometries are discontinous, such as the map that collapses the unit interval in $\mathbb{R}$ to the origin and then scales everything by a factor of two as well as the example illustrated above. These maps are also examples of a *pseudo-isometry*, a term introduced by Mostow [10] in 1974 in his study of arbitrary symmetric spaces (see also §5.9 of [11]). A pseudo-isometry satisfies a stronger condition than a quasi-isometry in that it has no additive term on the upper bound. Indeed, a pseudo-isometry is a Lipschitz map that distorts distances by a uniformly bounded amount, above a given scale. Not all continuous quasi-isometries are pseudo-isometries. An explicit example is given in Example 4.2.

Our main result is to show that a surgery specified by a pseudo-isometry yields a natural map from the original space to surgered space that is also a pseudo-isometry.

Theorem 1.1. *Suppose (X, d_X) and (T, d_T) are metric spaces and consider a subset $S \subset X$ as a metric space with metric induced from X. If $f\colon\ S \to T$ is a pseudo-isometry, then the natural map $F\colon\ X \to \widehat{X}_f$ is a pseudo-isometry as well.*

The definition of $\widehat{X}_f$ appears in Section 2. The proof of Theorem 1.1 appears in Section 3.

The metric on $\widehat{X}_f$ is defined via certain alternating sequences of pairs of points in X and T taking into account the function f. We call such sequences *admissible* and the length of such is the sum of the distances for each pair (Definition 2.3). The main technical step to prove Theorem 1.1 is Lemma 2.5 where we give a lower bound on the length of an admissible sequence in terms of the distance in X between its endpoints. It is in this lemma that we need to restrict to pseudo-isometries as opposed to quasi-isometries to control the amount of additive error.

One example that is covered by Theorem 1.1 is the map $F\colon\ \mathbb{R} \to \mathbb{R}$ that collapses each interval of the form $[2n, 2n+1]$ to a point. This is the example discussed above. For this example we have that $S = \{[2n, 2n+1] \mid n \in \mathbb{Z}\}$, $T = \mathbb{Z}$ and $f(s) = \lfloor \frac{s}{2} \rfloor$. In this case, the surgered space is again $\mathbb{R}$.

An application of Theorem 1.1 to regular trees is given in Example 4.1.

In Example 4.2 we show the "pseudo-" assumption is necessary in the following sense: When the gluing map is weakened to that of a quasi-isometry the natural map to the surgered space fails to even be a quasi-isometry. It remains open under what conditions does a quasi-isometric gluing map yield a quasi-isometric natural map between the original and the surgered space.

2. CONSTRUCTION OF THE SURGERED SPACE.

In this Section we define the surgered space $\widehat{X}_f$ using a notion of *admissible sequences* (Definition 2.3) which ties together the spaces X, S, and T via the pseudo-isometry $f\colon\ S \to T$. We also present a few properties of admissible sequences that form the essential parts of the proof of Theorem 1.1.

To begin, we state the definition of a pseudo-isometry.

Definition 2.1. Let (S, d_S) and (T, d_T) be metric spaces. A map $f\colon\ S \to T$ is a *pseudo-isometry* if there exist contants $K \geq 1$ and $C \geq 0$ such that the following hold.

1. For all $x_0, x_1 \in S$, we have:

$$\frac{1}{K} d_S(x_0, x_1) - C \leq d_T(f(x_0), f(x_1)) \leq K d_S(x_0, x_1).$$

2. For all $y \in T$, there is an $x \in S$ with $d_T(f(x), y) \leq C$.

Remark 2.2. If we allow the upper bound to also have an additive constant, i.e.,

$$d_T(f(x_0), f(x_1)) \leq K d_S(x_0, x_1) + C$$

then the map is called a *quasi-isometry*.

A metric on the surgered space will be defined via sequences of pairs of points that (potentially) intersect the sets where the surgery occurs, called *admissible sequences*. In what follows X and T are metric spaces with metrics d_X and d_T respectively, and $S \subseteq X$ is a subspace considered as a metric space with the metric induced from X. We also have a pseudo-isometry $f\colon\ S \to T$ with constants K and C as in Definition 2.1.

Definition 2.3. An *admissible sequence* is a sequence of pairs of the form:

$$\gamma : (x_0, y_1), (u_1, v_1), (x_1, y_2), \ldots, (u_k, v_k), (x_k, y_{k+1}) \tag{2.1}$$

where:

1. $x_0, y_{k+1} \in X$,
2. $x_i, y_i \in S$ for $i = 1, \ldots, k$,
3. $u_i, v_i \in T$ for $i = 1, \ldots, k$, and
4. $u_i = f(y_i)$ and $v_i = f(x_i)$ for $i = 1, \ldots, k$.

We allow for the possibility that $x_i = y_{i+1}$ or $u_i = v_i$ for each $i = 0, \ldots, k$. Moreover, we allow for the possibility that the pair (x_0, y_1) is omitted. In this case, the only restriction on u_1 is that it lies in T. Likewise, we allow for the possibility that the pair (x_k, y_{k+1}) is omitted. In this case, the only restriction on v_k is that it lies in T. We say the sequence is from x_0 to y_{k+1}, modifying to use u_1 or v_k accordingly if the pair (x_0, y_1) or (x_k, y_{k+1}) respectively is omitted. A schematic for an admissible sequence appears in Figure 2.

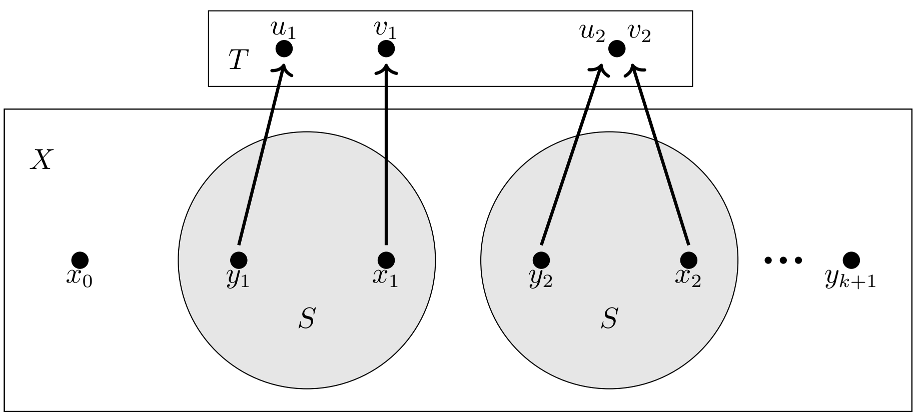

Figure 3. An admissible sequence

Definition 2.4. The *length* of an admissible sequence γ as defined in (2.1) is:

$$\ell(\gamma) = d_X(x_0, y_1) + \sum_{i=1}^{k} \big(d_T(u_i, v_i) + d_X(x_i, y_{i+1})\big). \tag{2.2}$$

The next lemma shows that the length of an admissible sequence between points x and y in X is bounded below by a linear function of the distance in X between x and y. This lemma is essential to the proof of Theorem 1.1 as it forms the basis of the proving the pseudo-isometry inequalities.

Lemma 2.5. *Let γ be an admissible sequence from x to y, where $x, y \in X$. Then*

$$d_X(x,y) \leq K^2\ell(\gamma) + KC.$$

Proof. Let the admissible sequence γ be given by:

$$\gamma : (x_0, y_1), (u_1, v_1), (x_1, y_2), \ldots, (u_k, v_k), (x_k, y_{k+1})$$

where $x_0 = x$ and $y_{k+1} = y$.

By the definition of an admissible sequence, we have $f(x_i) = v_i$ and $f(y_i) = u_i$ for $i = 1, \ldots, k$. The assumption that f is a pseudo-isomtery implies

$$d_X(y_1, x_k) \leq K d_T(u_1, v_k) + KC \text{ and } d_T(v_i, u_{i+1}) \leq K d_X(x_i, y_{i+1}).$$

Combining the triangle inequality with the first of these inequalities of the gives (2.3), (2.4) and (2.5). Regrouping gives (2.6) and the second of these inequalities is used in (2.7). As $K \geq 1$, further rearranging gives (2.8). Finally, (2.9) follows from the definition of the length of γ.

$$\begin{aligned}
d_X(x,y) &\leq d_X(x_0,y_1) + d_X(y_1,x_k) + d_X(x_k,y_{k+1}) && (2.3)\\
&\leq d_X(x_0,y_1) + (K d_T(u_1,v_k) + KC) + d_X(x_k,y_{k+1}) && (2.4)\\
&\leq d_X(x_0,y_1) + K\left(\sum_{i=1}^{k-1}\big(d_T(u_i,v_i) + d_T(v_i,u_{i+1})\big) + d_T(u_k,v_k)\right)\\
&\qquad + d_X(x_k,y_{k+1}) + KC && (2.5)\\
&= d_X(x_0,y_1) + K\left(\sum_{i=1}^{k} d_T(u_i,v_i) + \sum_{i=1}^{k-1} d_T(v_i,u_{i+1})\right)\\
&\qquad + d_X(x_k,y_{k+1}) + KC && (2.6)\\
&\leq d_X(x_0,y_1) + K\left(\sum_{i=1}^{k} d_T(u_i,v_i) + K\sum_{i=1}^{k-1} d_X(x_i,y_{i+1})\right)\\
&\qquad + d_X(x_k,y_{k+1}) + KC && (2.7)\\
&\leq K^2\left(d_X(x_0,y_1) + \sum_{i=1}^{k}\big(d_T(u_i,v_i) + d_X(x_i,y_{i+1})\big)\right) + KC && (2.8)\\
&\leq K^2\ell(\gamma) + KC. && (2.9)
\end{aligned}$$

This completes the proof of the lemma. ∎

Remark 2.6. Note that in the proof above it is necessary that f is a pseudo-isometry and not merely a quasi-isometry. In passing from (2.6) to (2.7), the d_X summation has no additive term, effectively allowing us to bound the lengths with a multiplicative constant. Had f been just a quasi-isometry this summation would induce $k-1$ additive constants. The number of such constants reflects the number of steps in the admissible sequence which is not bounded by the distance. This makes it impossible to bound the distance between x and y in terms of the length of an admissible sequence between them.

As we complete the construction of the surgered space let us recall the orginal space X, a subset $S \subset X$ and a pseudo-isometry $f\colon\ S \to T$. We first glue S to T forming the space X':

$$X' = {X \cup T}\Big/{s \sim f(s),\ \forall s \in S}.$$

In other words, points in X' are equivalence classes. Let $j\colon\ X \cup T \to X'$ the quotient map that takes a point to its equivalence class. These equivalence classes are one of three types:

1. If $x \in X - S$, then $j(x) = \{x\}$, a singleton,
2. If $x \in S$, then $j(x) = \{y \in S \mid f(y) = f(x)\} \cup \{f(x)\}$, or
3. If $u \in T$, then $j(u) = \{y \in S \mid f(y) = u\} \cup \{u\}$.

Note, the first set in the union for type 3. may be empty.

We obtain a pseudo-metric on X' as follows. The infimum of lengths of admissible sequences induces a pseudo-metric $p_{X'}\colon\ X' \times X' \to \mathbb{R}$ defined by:

$$p_{X'}(x', y') = \inf\{\ell(\gamma) \mid \gamma \text{ is an admissible sequence from } x \text{ to } y \text{ where } j(x) = x' \text{ and } j(y) = y'\}.$$

We define $(\widehat{X}_f, d_{\widehat{X}_f})$ as the metric space induced by identifying points in $(X', p_{X'})$ that have pseudo-distance equal to 0. If the corresponding quotient map is $q\colon\ X' \to \widehat{X}_f$ we have

$$d_{\widehat{X}_f}(\hat{x}, \hat{y}) = \inf\{p_{X'}(x', y') \mid q(x') = \hat{x} \text{ and } q(y') = \hat{y}\}.$$

There is an induced map $F\colon\ X \to \widehat{X}_f$ given by the composition:

$$F\colon\ X \xrightarrow{j} X' \xrightarrow{q} \widehat{X}_f.$$

Summarizing the above, we have that $d_{\widehat{X}_f}(\hat{x}, \hat{y})$ is the infimum of the set of lengths of admissible sequences from a point in $F^{-1}(\hat{x})$ to a point in $F^{-1}(\hat{y})$. The map F can be thought of as a kind of surgery on X, in which a subset S is exised and replaced by a set T.

We remark here that an immediate consequence of this definition is that

$$d_{\widehat{X}_f}(q(x'), q(y')) \leq p_{X'}(x', y')\ \forall\, x', y' \in X'.$$

This will be used in the proof of Theorem 1.1.

The lemma below indicates that if this surgery glues two points together then the two points were a bounded distance apart in the original metric.

Lemma 2.7. *If $x, y \in X$ and $F(x) = F(y)$, then $d_X(x, y) \leq 3KC$.*

Proof. Fix points $x, y \in X$ and suppose that $F(x) = F(y) = \hat{x}$. Let $x' = j(x)$ and $y' = j(y)$. By the construction of $\widehat{X}_f$, we have that for any two points in $q^{-1}(\hat{x})$ the pseudo-distance is equal to 0. Hence $p_{X'}(x', y') = 0$. Therefore, for any $\epsilon > 0$, there must be points $x_0, y_0 \in X \cup T$ with $j(x_0) = x'$, $j(y_0) = y'$, and an admissible sequence γ from x_0 to y_0 of length less than ϵ.

By the definition of $j\colon\ X \cup T \to X'$ we can assume that x_0 and y_0 lie in X. Indeed, if $x_0 \in T$ then we must have $x_0 = f(s_0)$ for some s_0 as j is injective on

$T - f(S)$. We could then prepend the sequence γ by the ordered pair (s_0, s_0) to get a new sequence with the same properties that starts in X. Likewise if $y_0 \in T$.

Since $j(x) = j(x_0)$, we must have that $f(x) = f(x_0)$ and hence $d_X(x, x_0) \leq KC$ since f is a pseudo-isometry. Likewise we have that $d_X(y, y_0) \leq KC$. Using Lemma 2.5, we have that $d_X(x_0, y_0) \leq K^2\epsilon + KC$. Combining these with the triangle inequality, we find:

$$d_X(x,y) \leq d_X(x,x_0) + d_X(x_0,y_0) + d_X(y_0,y) \leq KC + K^2\epsilon + KC + KC.$$

As this holds for all $\epsilon > 0$, we have $d_X(x,y) \leq 3KC$ as claimed. ■

3. PROOF OF MAIN THEOREM We can now prove the main theorem. It is restated here for convenience.

Theorem 1.1. *Suppose (X, d_X) and (T, d_T) are metric spaces and consider a subset $S \subset X$ as a metric space with metric induced from X. If $f\colon\ S \to T$ is a pseudo-isometry, then the natural map $F\colon\ X \to \widehat{X}_f$ is a pseudo-isometry as well.*

Proof. Let X, S, T be as in the statement of the theorem and $f\colon\ S \to T$ a pseudo-isometry with constants $K \geq 1$ and $C > 0$ as in Definition 2.1. Let X' and $\widehat{X}_f$ be as defined in Section 2 and $F\colon\ X \to \widehat{X}_f$ the map induced by the composition:

$$F\colon\ X \xrightarrow{j} X' \xrightarrow{q} \widehat{X}_f.$$

First we show F is coarsely surjective. By construction, for any $\hat{y} \in \widehat{X}_f - F(X)$ there is some $t \in T$ such that $\hat{y} = q(j(t))$. As $f\colon\ S \to T$ is coarsely surjective, there is an $s \in S$ where $d_T(f(s), t) \leq C$. The admissible sequence $(s, s), (f(s), t)$ from s to t has length at most C. Therefore $p_{X'}(j(s), j(t)) \leq C$ and hence

$$d_{\widehat{X}_f}(F(s), \hat{y}) = d_{\widehat{X}_f}(q(j(s)), q(j(t))) \leq p_{X'}(j(s), j(t)) \leq C$$

as well.

Next we demonstrate an upper bound on $d_{\widehat{X}_f}(F(x), F(y))$. Given points $x, y \in X$, for the admissible sequence $\gamma : (x, y)$ we find that $p_{X'}(j(x), j(y)) \leq \ell(\gamma) = d_X(x, y)$. Therefore

$$d_{\widehat{X}_f}(F(x), F(y)) = d_{\widehat{X}_f}(q(j(x)), q(j(y))) \leq p_{X'}(j(x), j(y)) \leq d_X(x, y).$$

Finally, we demonstrate a lower bound on $d_{\widehat{X}_f}(F(x), F(y))$. Let $\epsilon > 0$. Given points $x, y \in X$, we fix points $x_0, y_0 \in X$ where $F(x) = F(x_0)$, $F(y) = F(y_0)$, and for which there exists an admissible sequence γ from x_0 to y_0 of length less than $d_{\widehat{X}_f}(F(x), F(y)) + \epsilon$. We note that such points and admissible sequence exist by an argument similar to the one presented in Lemma 2.7.

By Lemmas 2.5 and 2.7 we now have that:

$$\begin{aligned} d_X(x,y) &\leq d_X(x,x_0) + d_X(x_0,y_0) + d_X(y_0,y) \\ &\leq 3KC + K^2\ell(\gamma) + KC + 3KC \\ &\leq K^2\left(d_{\widehat{X}_f}(F(x),F(y)) + \epsilon\right) + 7KC. \end{aligned}$$

As this holds for all $\epsilon > 0$, rearranging we find that:

$$\frac{1}{K^2} d_X(x,y) - \frac{7C}{K} \leq d_{\widehat{X}_f}(F(x), F(y)). \qquad \blacksquare$$

4. EXAMPLES In this section we present two examples. The first is an application of Theorem 1.1 where we show that the regular tree of degree 4 is pseudo-isometric to the regular tree of degree 6 (Example 4.1). The second example shows that the "pseudo-isometry" assumption in Theorem 1.1 is necessary even if the conclusion is also weakened (Example 4.2).

Example 4.1. Let X be the regular tree of degree 4 where every edge is isometric to the unit interval $[0,1]$. We consider the subset $S \subset X$ consisting of every other horizontal edge, including its incident vertices, as indicated in Figure 4. Specifically, we can identify X as the Cayley graph of the free group of rank 2, F_2, corresponding to a basis $\{g_1, g_2\}$ (see Chapter 2 of [8] for details). Then there are two orbits of edges the corresponding to the set of horizontal edges and the set of vertical edges respectively in Figure 4. Let e_1 be the edge whose originating vertex is the identity and whose terminal vertex is g_1. Then the set S corresponds to the set of edges of the form we_1 where the terminal syllable of w is an even power of g_1. In other words, $w \in F_2$ is of the form:

$$w = w' g_1^{2p}$$

where $w' \in F_2$ is either trivial or ends in $g_2^{\pm 1}$ and $p \in \mathbb{Z}$.

Let $T \subset X$ be the set of midpoints of the edges in S, considered as a metric space using the metric from X. The map $f\colon\ S \to T$ that sends each edge in S to its midpoint is a pseudo-isometry where $K = 2$ and $C = 1$. As shown in Figure 4, the surgered space $\widehat{X}_f$ is the regular tree of degree 6. This example can be generalized to construct explicit pseudo-isometries between regular trees with other degrees as well.

Example 4.2. Let X be closed positive ray in $\mathbb{R}$, i.e., $X = [0, \infty)$. Define two sequences $(a_i), (b_i) \subset X$ as follows starting with $i = 0$:

$$a_i = 1 + i - (1/2)^i : 0, 3/2, 11/4, 31/8, \ldots$$
$$b_i = 2 + i - (1/2)^i : 1, 5/2, 15/4, 39/8, \ldots$$

These sequences are related by the equations $a_i = b_{i-1} + (1/2)^i$ and $b_i = a_i + 1$. We consider the subset $S \subset X$ given by:

$$S = \bigcup_{i=0}^{\infty} [a_i, b_i].$$

See Figure 5.

Let $T \subset X$ be the set of all a_i, considered as a metric space using the metric from X. The map $f\colon\ S \to T$ that sends each interval $[a_i, b_i]$ to a_i is a (continuous) quasi-isometry with $K = 1, C = 2$, but it is not a pseudo-isometry for any choice of constants K and C. Indeed, $d_S(b_{i-1}, a_i) = (1/2)^i \to 0$ as $i \to \infty$ while $d_T(f(b_{i-1}), f(a_i)) = d_T(a_{i-1}, a_i) \geq 1$ for all i. In this case, the surgered space $\widehat{X}_f$ is isometric to the half-open interval $[0,1) \subset \mathbb{R}$. In particular, X is not pseudo-isometric (nor quasi-isometric) to $\widehat{X}_f$.

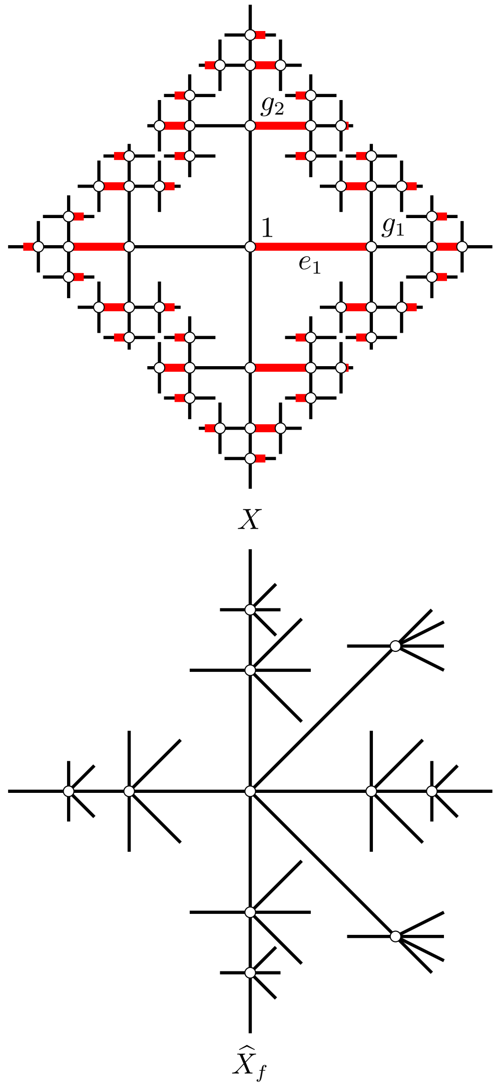

Figure 4. The regular tree of degree 4 with the subset S (shown in red). The surgered space is the regular tree of degree 6.

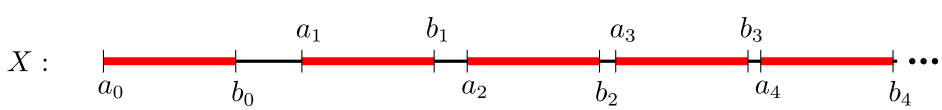

Figure 5. The closed positive ray with the subset S (shown in red). The surgered space is the half-open interval $[0, 1) \subset \mathbb{R}$.

The failure of this example is directly tied to the failure for Lemma 2.5 in this setting. There are admissible sequences with bounded length connecting points arbitrarily far apart in X. See Remark 2.6.

This example shows the necessity of the hypothesis of a pseudo-isometry in the statement of Theorem 1.1. It would be interesting to find robust conditions on $S \subset X$

so that if $f\colon\ S \to T$ is a quasi-isometry, then the surgered space $\widehat{X}_f$ is quasi-isometric to X.

REFERENCES

[1] Milnor J. A procedure for killing homotopy groups of differentiable manifolds. In: Proc. Sympos. Pure Math., Vol. III. Amer. Math. Soc., Providence, RI; 1961. p. 39-55.
[2] Mrowka TS, Ozsváth PS, editors. Low dimensional topology. vol. 15 of IAS/Park City Mathematics Series. American Mathematical Society, Providence, RI; 2009. Lecture notes from the 15th Park City Mathematics Institute (PCMI) Graduate Summer School held in Park City, UT, Summer 2006. Available from: https://doi.org/10.1090/pcms/015.
[3] Stillwell J. Poincaré and the early history of 3-manifolds. Bull Amer Math Soc (NS). 2012;49(4):555-76. Available from: https://doi.org/10.1090/S0273-0979-2012-01385-X.
[4] Rolfsen D. Knots and links. vol. 7 of Mathematics Lecture Series. Publish or Perish, Inc., Houston, TX; 1990. Corrected reprint of the 1976 original.
[5] Švarc AS. A volume invariant of coverings. Doklady Akademii Nauk SSSR. 1955;3:32 -34.
[6] Milnor J. A note on curvature and fundamental group. J Differential Geometry. 1968;2:1-7. Available from: http://projecteuclid.org/euclid.jdg/1214501132.
[7] Gromov M. Hyperbolic manifolds, groups and actions. In: Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978). vol. No. 97 of Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ; 1981. p. 183-213.
[8] Clay M, Margalit D. Office Hours with a Geometric Group Theorist. Princeton University Press; 2017.
[9] Bridson MR, Haefliger A. Metric spaces of non-positive curvature. vol. 319 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin; 1999. Available from: https://doi.org/10.1007/978-3-662-12494-9.
[10] Mostow GD. Strong rigidity of locally symmetric spaces. vol. No. 78 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo; 1973.
[11] Thurston WP. Three-dimensional geometry and topology. Vol. 1. vol. 35 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ; 1997.