- Applying power series to compute limits
- Taylor's Remainder: How to know when "n" is good enough

(ex) $$\lim_{x\to 0} \frac{\tan^{-1}(7x) - 7x\cos(7x) - \frac{343}{6}x^3}{x^5}$$

1. notice: evaluate the limit immediately $= 0/0$
2. plan: expand tan & cos into series (keep summation notation)
3. hint: pull out 1^{st} few terms (avoid div by 0)

$\tan^{-1}(7x)$:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \Rightarrow \tan^{-1}(7x) = \sum_{n=0}^{\infty} (-1)^n \frac{(7x)^{2n+1}}{2n+1} = 7x - \frac{(7x)^3}{3} + \frac{(7x)^5}{5} - \dots$$

$7x\cos(7x)$:

$$\cos(x) = \sum (-1)^n \frac{x^{2n}}{(2n)!}, \quad \cos(7x) = \sum_{n=1}^{\infty} (-1)^n \frac{(7x)^{2n}}{(2n)!}, \quad 7x\cos(7x) = 7x\sum_{n=0}^{\infty} (-1)^n \frac{(7x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(7x)^{2n}\,7x}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{7x^{2n+1}}{(2n)!}$$

$$= 7x - \frac{(7x)^3}{2} + \frac{(7x)^5}{4!} - \dots$$

$$= \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n (7x)^{2n+1}}{2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n (7x)^{2n+1}}{(2n)!} - \frac{343}{6}x^3}{x^5}$$

$$= \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} (-1)^n (7x)^{2n+1}\left[\frac{1}{2n+1} - \frac{1}{(2n)!}\right] - \frac{343}{6}x^3}{x^5}$$

$$\frac{x^{2n+1}}{x^5} = x^{2n-4}$$ greater than 5

$$= \lim_{x\to 0} \frac{\underbrace{7x(1-1)}_{(n=0)\ \to 0} - \underbrace{(7x)^3\left(\frac{1}{3}-\frac{1}{2}\right)}_{(n=1)\ = -1/6} + \underbrace{(7x)^5\left(\frac{1}{5}-\frac{1}{24}\right)}_{(n=2)} + \sum_{n=3}^{\infty} (-1)^n (7x)^{2n+1}\left[\frac{1}{2n+1} - \frac{1}{(2n)!}\right] - \frac{343}{6}x^3}{x^5}$$

$$= \lim_{x\to 0} \frac{7^5\left(\frac{1}{5}-\frac{1}{24}\right) + \sum_{n=3}^{\infty} (-1)^n (7)^{2n+1} x^{2n-4}\left[\frac{1}{2n+1} - \frac{1}{(2n)!}\right]}{} = 7^5 \cdot \frac{19}{120} = \frac{319333}{120}$$

take limit $x \to 0$

$$= \lim_{x\to 0} \frac{\frac{319333}{120} - 7^7 \cdot x^2 \cdot \left(\frac{1}{7} - \frac{1}{6!}\right) + \text{stuff}\cdot x^4 - \text{more stuff}\ x^6 + \dots}{}$$

$\to 0$ $\to 0$ $\to 0$ $\to 0$

$$7^5 \cancel{x^5} = \frac{(7x)^5\left(\frac{1}{120}\right)}{\cancel{x^5}}$$

Taylor's Remainder theorem

Recall the Taylor expansion for a function $f(x)$:

$$e^x \approx \overbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!}}^{T_3}$$

3rd order

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n$$

$$R_3(x) = \underbrace{\frac{x^4}{4!} + \frac{x^5}{5!} + \ldots}_{\text{the tail}}$$

The error between $f(x)$ & the $N^{\underline{th}}$ order series
(remainder)

$$\sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n$$ is $R_N(x)$ and

$$|R_N(x)| \leq \frac{M \cdot |x-c|^{N+1}}{(N+1)!}$$ where $M \geq f^{(N+1)}(x)$ for all x in the interval of convergence

the error incurred by approximating with only N terms is proportional to the next N+1 term in the series … the constant of proportionality M depends on how big the N+1 derivative is nearby

Estimate $e^{1.5}$ up to 10^{-3} accuracy on $[-2,2]$.

Since 1.5 is "small", we'll use MacLaurin! $c=0$

$$e^x = \sum_{n=0}^{N} \frac{x^n}{n!}$$

N ----> How big to get close to $e^{1.5}$

Know

$$|R_N(x)| \le \frac{M x^{N+1}}{(N+1)!}$$

Here $x = 1.5$ & M needs to be $\ge f^{N+1}(x)$ on $[-2,2]$

$\qquad = e^x$ on $[-2,2]$

$$\frac{9(1.5)^{N+1}}{(N+1)!} \le 10^{-3}$$

e^2 = largest val

$e \approx 2.7$, so $M = 9$ works $\quad 9 > e^x$ when $x \in [-2,2]$

calcular, experiment

... $\Rightarrow n = 5$ works!