ma516-hw1

p. 14, number 5

Let $\mathcal{A}$ be a collection of sets. Determine the truth of each of the following statements and of their converses.

a. $x \in \bigcup_{A \in \mathcal{A}} \implies x \in A$ for at least one $A \in \mathcal{A}$.

True. By definition of union, if x is in the union of the sets in $\mathcal{A}$, then there exists at least one set A in $\mathcal{A}$ such that x is in A. Conversely, if $x \in A$ for at least one $A \in \mathcal{A}$, then by definition of union, x must be in the union of the sets in $\mathcal{A}$. Thus, both the statement and its converse are true.

b. $x \in \bigcup_{A \in \mathcal{A}} \implies x \in A$ for every $A \in \mathcal{A}$.

False. For example, let $\mathcal{A} = \{\{1, 2\}, \{3, 4\}\}$. Then $x = 1$ is in $\bigcup_{A \in \mathcal{A}} = \{1, 2, 3, 4\}$, but 1 is not in every set in $\mathcal{A}$ (it is not in $\{3, 4\}$). Conversely, if $x \in A$ for every $A \in \mathcal{A}$, then x must be in the intersection of all sets in $\mathcal{A}$. Since the intersection is a subset of the union, x must also be in the union. Thus, the converse is true.

c. $x \in \bigcap_{A \in \mathcal{A}} \implies x \in A$ for at least one $A \in \mathcal{A}$.

True. By definition of intersection, if x is in the intersection of the sets in $\mathcal{A}$, then x is in every set A in $\mathcal{A}$. Therefore, x is certainly in at least one set A in $\mathcal{A}$. Conversely, if $x \in A$ for at least one $A \in \mathcal{A}$, it does not necessarily imply that x is in the intersection of all sets in $\mathcal{A}$. For example, let $\mathcal{A} = \{\{1, 2\}, \{2, 3\}\}$. Then $x = 1$ is in at least one set ($\{1, 2\}$), but it is not in the intersection ($\{2\}$). Thus, the converse is false.

d. $x \in \bigcap_{A \in \mathcal{A}} \implies x \in A$ for every $A \in \mathcal{A}$.

True. By definition of intersection, if x is in the intersection of the sets in $\mathcal{A}$, then x is in every set A in $\mathcal{A}$. Conversely, if $x \in A$ for every $A \in \mathcal{A}$, then by definition of intersection, x must be in the intersection of all sets in $\mathcal{A}$. Thus, both the statement and its converse are true.

p. 15, number 9

Formalize and prove DeMorgan's law for arbitrary unions and intersections.

DeMorgan's laws for arbitrary unions and intersections state the following:

1. The complement of the union of a collection of sets is equal to the intersection of their complements:
$$\left(\bigcup_{A \in \mathcal{A}} A \right)^c = \bigcap_{A \in \mathcal{A}} A^c$$
2. The complement of the intersection of a collection of sets is equal to the union of their complements:
$$\left(\bigcap_{A \in \mathcal{A}} A \right)^c = \bigcup_{A \in \mathcal{A}} A^c$$

   Proof of the first law: Let $x \in \left(\bigcup_{A \in \mathcal{A}} A \right)^c$. This means that x is not in the union of the sets in $\mathcal{A}$. Therefore, for every set A in $\mathcal{A}$, x is not in A. This implies that x is in the complement of each set A, i.e., $x \in A^c$ for every $A \in \mathcal{A}$. Hence, $x \in \bigcap_{A \in \mathcal{A}} A^c$. Conversely, let $x \in \bigcap_{A \in \mathcal{A}} A^c$. This means that for every set A in $\mathcal{A}$, x is in the complement of A, i.e., x is not in A. Therefore, x is not in the union of the sets in $\mathcal{A}$, which implies that $x \in \left(\bigcup_{A \in \mathcal{A}} A \right)^c$. Thus, we have shown that both sides are equal.

Proof of the second law: Let $x \in \left(\bigcap_{A \in \mathcal{A}} A \right)^c$. This means that x is not in the intersection of the sets in $\mathcal{A}$. Therefore, there exists at least one set A in $\mathcal{A}$ such that x is not in A. This implies that x is in the complement of that set A, i.e., $x \in A^c$ for some $A \in \mathcal{A}$. Hence, $x \in \bigcup_{A \in \mathcal{A}} A^c$. Conversely, let $x \in \bigcup_{A \in \mathcal{A}} A^c$. This means that there exists at least one set A in $\mathcal{A}$ such that x is in the complement of A, i.e., x is not in A. Therefore, x is not in the intersection of the sets in $\mathcal{A}$, which implies that $x \in \left(\bigcap_{A \in \mathcal{A}} A \right)^c$. Thus, we have shown that both sides are equal.

p. 20, number 1

Let $f : A \to B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$. Prove the following statements.

a. $A_0 \subseteq f^{-1}(f(A_0))$ and equality holds if f is injective. To prove that $A_0 \subseteq f^{-1}(f(A_0))$, let $x \in A_0$. By the definition of the image of a set under a function, $f(x) \in f(A_0)$. Therefore, by the definition of the preimage, $x \in f^{-1}(f(A_0))$. This shows that every element of A_0 is also in $f^{-1}(f(A_0))$, hence $A_0 \subseteq f^{-1}(f(A_0))$. If f is injective, but somehow equality does not hold, then there exists some $y \in f^{-1}(f(A_0))$ such that $y \notin A_0$. By the definition of preimage, this means that $f(y) \in f(A_0)$. Therefore there exists some $x \in A_0$ such that $f(y) = f(x)$. But since f is injective, this implies that $y = x$, which contradicts the assumption that $y \notin A_0$. Therefore, if f is injective, we must have equality: $A_0 = f^{-1}(f(A_0))$.

b. $f(f^{-1}(B_0)) \subseteq B_0$ and equality holds if f is surjective. To prove that $f(f^{-1}(B_0)) \subseteq B_0$, let $y \in f(f^{-1}(B_0))$. By the definition of the preimage, there exists some $x \in f^{-1}(B_0)$ such that $f(x) = y$. Since $x \in f^{-1}(B_0)$, by the definition of preimage, we have $f(x) \in B_0$. Therefore, $y \in B_0$. This shows that every element of $f(f^{-1}(B_0))$ is also in B_0, hence $f(f^{-1}(B_0)) \subseteq B_0$. If f is surjective, but somehow equality does not hold, then there exists some $y \in B_0$ such that $y \notin f(f^{-1}(B_0))$. Since f is surjective, there exists some $x \in A$ such that $f(x) = y$. Since $y \in B_0$, this implies that $x \in f^{-1}(B_0)$. Therefore, $y = f(x) \in f(f^{-1}(B_0))$, which contradicts the assumption that $y \notin f(f^{-1}(B_0))$. Therefore, if f is surjective, we must have equality: $f(f^{-1}(B_0)) = B_0$.

p. 39, number 4

Let $m, n \in \mathbb{Z}_+$ and let $X \neq \phi$.

a. If $m \leq n$, find an injective function $f : X^m \to X^n$. Define the function $f : X^m \to X^n$ by

$$f(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_m, x_1, x_1, \dots, x_1)$$

where we append $n - m$ copies of x_1 to the end of the tuple. This function is injective because if $f(x_1, x_2, \dots, x_m) = f(y_1, y_2, \dots, y_m)$, then the first m components must be equal, which implies that $(x_1, x_2, \dots, x_m) = (y_1, y_2, \dots, y_m)$.

b. Find a bijective map $g : X^m \times X^n \to X^{m+n}$. Define the function $g : X^m \times X^n \to X^{m+n}$ by

$$g((x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_n)) = (x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$$

This function is bijective because it is both injective and surjective. It is injective because if $g((x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_n)) = g((u_1, u_2, \dots, u_m), (v_1, v_2, \dots, v_n))$, then the first m components must be equal and the last n components must be equal, which implies that $(x_1, x_2, \dots, x_m) = (u_1, u_2, \dots, u_m)$ and $(y_1, y_2, \dots, y_n) = (v_1, v_2, \dots, v_n)$. It is surjective because for any $(z_1, z_2, \dots, z_{m+n}) \in X^{m+n}$, we can write it as $g((z_1, z_2, \dots, z_m), (z_{m+1}, z_{m+2}, \dots, z_{m+n}))$.

c. Find an injective map $h : X^n \to X^\omega$ We append infinitely many copies of the first element to the end of the tuple. Define the function $h : X^n \to X^\omega$ by

$$h(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n, x_1, x_1, x_1, \dots)$$

where we append infinitely many copies of x_1 to the end of the tuple. This function is injective because if $h(x_1, x_2, \dots, x_n) = h(y_1, y_2, \dots, y_n)$, then the first n components must be equal, which implies that $(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$.

d. Find a bijective map $k : X^n \times X^\omega \to X^\omega$ We concatenate the finite sequence with the infinite sequence to form a new infinite sequence. Define the function $k : X^n \times X^\omega \to X^\omega$ by

$$k((x_1, x_2, \dots, x_n), (y_1, y_2, \dots)) = (x_1, x_2, \dots, x_n, y_1, y_2, \dots)$$

This function is bijective because it is both injective and surjective. It is injective because if $k((x_1, x_2, \dots, x_n), (y_1, y_2, \dots)) = k((u_1, u_2, \dots, u_n), (v_1, v_2, \dots))$, then the first n components must be equal and the remaining components must be equal, which implies that $(x_1, x_2, \dots, x_n) = (u_1, u_2, \dots, u_n)$ and $(y_1, y_2, \dots) = (v_1, v_2, \dots)$. It is surjective because for any $(z_1, z_2, \dots) \in X^\omega$, we can write it as $k((z_1, z_2, \dots, z_n), (z_{n+1}, z_{n+2}, \dots))$.

e. Find a bijective map $m : X^\omega \times X^\omega \to X^\omega$ We interleave the two sequences to form a new sequence. Define the function $m : X^\omega \times X^\omega \to X^\omega$ by

$$m((x_1, x_2, \dots), (y_1, y_2, \dots)) = (x_1, y_1, x_2, y_2, \dots)$$

This function is bijective because it is both injective and surjective. It is injective because if $m((x_1, x_2, \dots), (y_1, y_2, \dots)) = m((u_1, u_2, \dots), (v_1, v_2, \dots))$, then the odd-indexed components must be equal and the even-indexed components must be equal, which implies that $(x_1, x_2, \dots) = (u_1, u_2, \dots)$ and $(y_1, y_2, \dots) = (v_1, v_2, \dots)$. It is surjective because for any $(z_1, z_2, \dots) \in X^\omega$, we can write it as $m((z_1, z_3, z_5, \dots), (z_2, z_4, z_6, \dots))$.

f. If $A \subseteq B$, show that there is a injective map from $m : (A^\omega)^n \to B^\omega$. Define the function $m : (A^\omega)^n \to B^\omega$ by

$$m((a_{11}, a_{12}, \dots), (a_{21}, a_{22}, \dots), \dots, (a_{n1}, a_{n2}, \dots)) = (a_{11}, a_{21}, \dots, a_{n1}, a_{12}, a_{22}, \dots, a_{n2}, \dots)$$

This function is injective because if $m((a_{11}, a_{12}, \dots), (a_{21}, a_{22}, \dots), \dots, (a_{n1}, a_{n2}, \dots)) = m((b_{11}, b_{12}, \dots), (b_{21}, b_{22}, \dots), \dots, (b_{n1}, b_{n2}, \dots))$, then the components corresponding to each A^ω must be equal, which implies that $(a_{i1}, a_{i2}, \dots) = (b_{i1}, b_{i2}, \dots)$ for each $i = 1, 2, \dots, n$. Thus, we have shown that m is injective.

P. 51, number 1

Show that $\mathbb{Q}$ is countable. To show that the set of rational numbers $\mathbb{Q}$ is countable, we can construct a bijection between $\mathbb{Q}$ and the set of natural numbers $\mathbb{N}$. We can represent each rational number as a fraction $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$. We can arrange these fractions in a two-dimensional grid, where the rows correspond to the numerator p and the columns correspond to the denominator q. We can then traverse this grid in a diagonal manner, starting from the fraction $\frac{0}{1}$ and moving to $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{1}$, $\frac{0}{2}$, $\frac{-1}{1}$, and so on. By doing this, we can list all the rational numbers in a sequence, which shows that there is a bijection between $\mathbb{Q}$ and $\mathbb{N}$. Therefore, $\mathbb{Q}$ is countable.

P. 83, 1

Let X be a topological space, let $A \subseteq X$. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

We can prove this by appealling to the definition of a topology, and to what it means for a set to be open. A set is open if it is a member of the topology. A topology is closed under arbitrary unions. By the given condition, for each $x \in A$, there is an open set U_x containing x such that $U_x \subseteq A$. Therefore, we can express A as the union of all such open sets:

$$A = \bigcup_{x \in A} U_x$$

Since each U_x is open and the union of open sets is also open, it follows that A is open in X.

Alternatively, we can prove this in terms of the Union Lemma (which says that open sets are union of basis elements). In the given condition, each U is an open set and therefore can be expressed as a union of basis elements. Thus each $x \in A$ is contained in a union of (a union of) basis elements, each of which is contained in A. Therefore, A can be expressed as a union of basis elements, and by the Union Lemma, A is open in X.