

# ma516-hw2-solutions

## Complete the following:

1. Let  $D$  be the set of dogs in Marquette, MI at this instant. Let  $a \sim b$  if dog  $a$  has the same mother as dog  $b$ . Show that  $\sim$  is an equivalence relation on  $D$ , and describe the equivalence classes.

**proof:** We need to show that  $\sim$  is reflexive, symmetric, and transitive. - Reflexive: For any dog  $a \in D$ ,  $a$  has the same mother as itself, so  $a \sim a$ . - Symmetric: If  $a \sim b$ , then dog  $a$  has the same mother as dog  $b$ . This means that dog  $b$  also has the same mother as dog  $a$ , so  $b \sim a$ . - Transitive: If  $a \sim b$  and  $b \sim c$ , then dog  $a$  has the same mother as dog  $b$ , and dog  $b$  has the same mother as dog  $c$ . This means that dog  $a$  has the same mother as dog  $c$ , so  $a \sim c$ .

2. Prove that a topology  $\mathcal{T}$  on a set  $X$  is the discrete topology if and only if  $\{x\} \in \mathcal{T}$  for all  $x \in X$ .

**proof:** ( $\Rightarrow$ ) If  $\mathcal{T}$  is the discrete topology on  $X$ , then by definition, every subset of  $X$  is an open set. In particular, for each  $x \in X$ , the singleton set  $\{x\}$  is a subset of  $X$ , and therefore  $\{x\} \in \mathcal{T}$ .

( $\Leftarrow$ ) Conversely, suppose that  $\{x\} \in \mathcal{T}$  for all  $x \in X$ . We want to show that  $\mathcal{T}$  is the discrete topology, which means that every subset of  $X$  is an open set. Let  $A$  be any subset of  $X$ . We can express  $A$  as a union of singleton sets:

$$A = \bigcup_{x \in A} \{x\}.$$

Since  $\{x\} \in \mathcal{T}$  for all  $x \in A$ , and  $\mathcal{T}$  is closed under arbitrary unions, it follows that  $A \in \mathcal{T}$ . Therefore, every subset of  $X$  is an open set, and  $\mathcal{T}$  is the discrete topology.

3. Define a topology on  $\mathbb{R}$  (by listing the open sets within it) that contains the open sets  $(0,2)$  and  $(1,3)$  and that contains as few open sets as possible.

**solution:** The smallest topology on  $\mathbb{R}$  that contains the open sets  $(0,2)$  and  $(1,3)$  is the topology generated by these two sets. This means that we need to include all finite intersections and arbitrary unions of these sets. The open sets in this topology are: - The empty set  $\emptyset$  (the intersection of no sets). - The sets  $(0,2)$  and  $(1,3)$  themselves. - The intersection of  $(0,2)$  and  $(1,3)$ , which is  $(1,2)$ . - The union of  $(0,2)$  and  $(1,3)$ , which is  $(0,3)$ .

All other unions and intersections of these sets will either be empty,  $(0,2)$ ,  $(1,3)$ ,  $(1,2)$ , or  $(0,3)$ . Therefore, the open sets in this topology are:

$$\mathcal{T} = \{\emptyset, (0, 2), (1, 3), (1, 2), (0, 3)\}.$$

4. For each  $n \in \mathbb{Z}$ , define

$$B(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n - 1, n, n + 1 & \text{if } n \text{ is even} \end{cases}$$

Show that the collection  $\mathcal{B} = \{B(n) : n \in \mathbb{Z}\}$  is a basis for a topology on  $\mathbb{Z}$ . Describe the open sets in this topology. The topology is called the *digital line topology*.

**proof:** To show that  $\mathcal{B}$  is a basis for a topology on  $\mathbb{Z}$ , we need to verify two conditions: 1. For each  $x \in \mathbb{Z}$ , there is at least one basis element  $B(n)$  such that  $x \in B(n)$ . 2. If  $x \in B(n_1) \cap B(n_2)$  for some  $n_1, n_2 \in \mathbb{Z}$ , then there is a basis element  $B(n_3)$  such that  $x \in B(n_3) \subseteq B(n_1) \cap B(n_2)$ . - For the first condition, if  $x$  is odd, then  $B(x) = \{x\}$  contains  $x$ . If  $x$  is even, then  $B(x) = \{x - 1, x, x + 1\}$  contains  $x$ . Therefore, for every integer  $x$ , there is a basis element that contains it. - For the second condition, suppose  $x \in B(n_1) \cap B(n_2)$ . If  $n_1$  and  $n_2$  are both odd, then  $B(n_1) = \{n_1\}$  and  $B(n_2) = \{n_2\}$ , so  $x$  must be equal to both  $n_1$  and  $n_2$ , which means  $n_1 = n_2 = x$ . In this case, we can take  $B(n_3) = B(x) = \{x\}$ , which is a subset of both  $B(n_1)$  and  $B(n_2)$ . If one of  $n_1$  or  $n_2$  is even, say  $n_1$  is even, then  $B(n_1) = \{n_1 - 1, n_1, n_1 + 1\}$ . If  $x$  is in this set, then  $x$  must be either  $n_1 - 1$ ,  $n_1$ , or  $n_1 + 1$ . If  $x$  is in  $B(n_2)$ , then  $x$  must be equal to  $n_2$  if  $n_2$  is odd, or  $x$  must be in  $\{n_2 - 1, n_2, n_2 + 1\}$  if  $n_2$  is even. In either case, we can find a basis element  $B(n_3)$  that contains  $x$  and is a subset of both  $B(n_1)$  and  $B(n_2)$ . Therefore,  $\mathcal{B}$  satisfies the conditions for being a basis for a topology on  $\mathbb{Z}$ .

The open sets in this topology are unions of the basis elements. Specifically, the open sets are: - The empty set  $\emptyset$ . - The singleton sets  $\{n\}$  for odd integers  $n$ . - The sets  $\{n - 1, n, n + 1\}$  for even integers  $n$ . - Unions of these sets, which can include combinations of odd and even integers. For example, the set  $\{1\} \cup \{2, 3, 4\}$  is an open set, as is the set  $\{1, 3\} \cup \{4, 5, 6\}$ .

5. Let  $S$  denote the set of points

$$\{(0, \frac{1}{n}) \in \mathbb{R}^2 : n \in \mathbb{Z}^+\}.$$

(Note that  $S$  is a set of points in  $\mathbb{R}^2$ , not open intervals.) Either prove that  $\mathbb{R}^2 - S$  is an open set in the standard topology on  $\mathbb{R}^2$  or prove that it is not an open set.

**proof:** To determine whether  $\mathbb{R}^2 - S$  is an open set, we need to check if for every point  $(x, y) \in \mathbb{R}^2 - S$ , there exists an open ball centered at  $(x, y)$  that is entirely contained within  $\mathbb{R}^2 - S$ . Since  $(0, 0)$  is not in  $S$ , we can consider the point  $(0, 0)$ . For any open ball centered at  $(0, 0)$  with radius  $r > 0$ , there will be points of the form  $(0, \frac{1}{n})$  for sufficiently large  $n$  that lie within this ball. This is because as  $n$  increases,  $\frac{1}{n}$  approaches 0, and thus points of the form  $(0, \frac{1}{n})$  will be arbitrarily close to  $(0, 0)$ . Therefore, there is no open ball centered at  $(0, 0)$  that is entirely contained within  $\mathbb{R}^2 - S$ . Hence,  $\mathbb{R}^2 - S$  is not an open set in the standard topology on  $\mathbb{R}^2$ .

Repeat the question for the set  $T = S \cup \{(0, 0)\}$ . Let  $x \in \mathbb{R}^2 - T$ . If  $x$  is not of the form  $(0, y)$  for some  $y \in \mathbb{R}$ , then we can find an open ball centered at  $x$  that does not intersect  $T$ , since  $T$  only contains points on the vertical line  $x = 0$ . If  $x$  is of the form  $(0, y)$  for some  $y \neq 0$ , then we can also find an open ball centered at  $x$  that does not intersect  $T$ , since the points in  $T$  are isolated and do not accumulate at any point other than  $(0, 0)$ . Therefore, for every point in  $\mathbb{R}^2 - T$ , there exists an open ball centered at that point that is entirely contained within  $\mathbb{R}^2 - T$ . Hence,  $\mathbb{R}^2 - T$  is an open set in the standard topology on  $\mathbb{R}^2$ . Incidentally, this shows that the set  $T$  is closed, since its complement is open.